

PAPER • OPEN ACCESS

An upper critical dimension for dynamo action: a d -dimensional closure model study

To cite this article: Sugan Durai Murugan *et al* 2026 *J. Phys. A: Math. Theor.* **59** 215702

View the [article online](#) for updates and enhancements.

You may also like

- [Integral formula for the propagator of the one-dimensional Hubbard model](#)
Taiki Ishiyama, Kazuya Fujimoto and Tomohiro Sasamoto
- [Continuum limits of lazy open quantum walks](#)
Lara Janiurek and Viv Kendon
- [On the complexity of quantum states and circuits from the orthogonal and symplectic groups](#)
Oxana Shaya, Zoë Holmes, Christoph Hirche et al.



PAPER

OPEN ACCESS

RECEIVED
11 January 2026REVISED
22 April 2026ACCEPTED FOR PUBLICATION
13 May 2026PUBLISHED
27 May 2026Original content from
this work may be used
under the terms of the
Creative Commons
Attribution 4.0 licence.Any further distribution
of this work must
maintain attribution to
the author(s) and the title
of the work, journal
citation and DOI.An upper critical dimension for dynamo action: a d -dimensional closure model studySugan Durai Murugan^{1,*} , Giorgio Krstulovic² , Dario Vincenzi³ and Samriddhi Sankar Ray⁴ ¹ Department of Mechanical Engineering, Johns Hopkins University, Baltimore, MD 21218, United States of America² Université Côte d'Azur, CNRS, Institut de Physique de Nice (INPHYNI), 17 rue Julien Lauprêtre, 06200 Nice, France³ Université Côte d'Azur, CNRS, LJAD, 06100 Nice, France⁴ International Centre for Theoretical Sciences, Tata Institute of Fundamental Research, Bengaluru 560089, India

* Author to whom any correspondence should be addressed.

E-mail: vsdmfriend@gmail.com, Giorgio.krstulovic@univ-cotedazur.fr, dario.vincenzi@univ-cotedazur.fr
and samriddhisankarray@gmail.com**Keywords:** dynamo, MHD turbulence, EDQNM closure, critical dimensions**Abstract**

We construct a d -dimensional eddy damped quasi-normal Markovian (EDQNM) closure model to study dynamo action in arbitrary dimensions. In particular, we find lower d_L and upper d_U critical dimensions for sustained dynamo action in this incompressible problem. Our model is adaptable for future studies incorporating helicity, compressible effects and a wide range of magnetic Reynolds and Prandtl numbers.

1. Introduction

Large magnetic fields are at the heart of almost every observation in astrophysics; indeed, they play a pivotal role in, as well as shape the consequence of, the dynamics of phenomena ranging from star formation, the interstellar medium to the underpinnings of the solar wind [1–3]. And yet questions remain about how such sustained magnetic fields arise—the *dynamo* problem—in the first place [4–6]. Since astrophysical flows are also, typically, notoriously turbulent, it is natural to look for answers to such questions within the framework of magnetohydrodynamic (MHD) turbulence [7–11]. While a theory for the dynamo problem rooted in the full set of equations for MHD is desirable, there are formidable challenges to this. From the point of view of direct numerical simulations (DNSs) of such systems, the parameter space accessible to modern simulations are quite far from what is realizable in either astrophysical systems or liquid-metal experiments [12]. For example, the Prandtl number, defined as the ratio of the kinetic viscosity to the magnetic diffusivity $Pm \equiv \nu/\eta$, ranges from values as large as 10^{14} (interstellar medium) to those as small as 10^{-5} (liquid sodium experiments). Such a range of numbers are prohibitively expensive for DNSs; thus more often than not, theoretical approaches based on reasonable assumptions provide additional insights and a fresh perspective in understanding the nuances of the dynamo problem.

An excellent example of such theoretical approaches, and the deep insights they provide, is the Kazantsev model for the fluctuation dynamo [13]. In this stochastic model, the velocity field is Gaussian and statistically homogenous, isotropic, and parity invariant. In addition, the correlation time is assumed to be zero—probably the strongest simplification in this model. By varying the features of the spatial correlations of the velocity field, it is possible to study the magnetic growth as a function of the degree of compressibility of the flow, its spatial regularity, the space dimension, and the Prandtl and magnetic Reynolds numbers (see, e.g., [2, 5, 6, 14, 15]). In particular, the Kazantsev model has provided the first evidence of the existence of a maximum critical dimension for the dynamo effect beyond which this random flow becomes unable to amplify a magnetic field [16, 17]. The range of dimensions where there is dynamo shrinks as the velocity becomes less and less regular in space, until it vanishes when the Hölder exponent of the velocity falls below 1/2 [18]. Compressibility, however, has the effect of widening the range of dimensions over which the dynamo is possible in this model [19]. Interestingly, dimension

$d = 3$ is the one where the least flow regularity is required for the dynamo effect to take place, independently of the degree of compressibility.

It is easy to appreciate why theoretical models with variable roughness (of the velocity field) and compressibility have a direct bearing on understanding *real* dynamos. Nevertheless and especially given the strong parallels of this problem to critical phenomena and phase transitions, the role of dimensions in the dynamo—no-dynamo transition deserves some attention. Taking this point of view and recalling the fundamental discoveries—such as dimensional regularization or the $4 - \epsilon$ expansion [20]—made possible by going beyond the physically obvious $d = 2$ or 3 dimensions, it is not unreasonable to ask if there is an analogue of a *lower* and *upper critical dimension* below and beyond which, respectively, dynamo action ceases to be. Indeed, such a point of view, of going beyond physically realizable integer dimensions of two and three, has led to interesting results on intermittency and energy cascades in classical fluid turbulence [21–27]. In this paper, we simply ask if there are lower d_L and upper d_U *critical* dimension within which dynamo action is confined?

While it is desirable to overcome the assumptions of Gaussianity and temporal decorrelation of the Kazantsev model and at the same time consider the fully nonlinear regime, it is difficult to answer the above question through DNSs in arbitrary dimension d . Instead, we construct a d -dimensional closure model for MHD turbulence, which in the absence of a magnetic field, reduces to the well-known eddy-damped quasi-normal Markovian (EDQNM) for fluid turbulence [21, 28–32]. We then perform detailed numerical simulations to show that for a given magnetic (Rm) and kinetic (Re) Reynolds number the dynamo action is constrained for dimensions $d_L \leq d \leq d_U$, with the lower critical dimension d_L marginally higher than 2 and a finite upper critical dimension d_U beyond which the dynamo cannot be sustained.

2. A d -dimensional closure model for MHD turbulence

The starting point of our study is the incompressible MHD equations

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla_p + (\mathbf{B} \cdot \nabla) \mathbf{B} + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \quad (1a)$$

$$\partial_t \mathbf{B} + (\mathbf{u} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{u} + \eta \nabla^2 \mathbf{B}, \quad (1b)$$

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{B} = 0. \quad (1c)$$

The magnetic field $\mathbf{B}(\mathbf{x}, t)$ and the velocity field $\mathbf{u}(\mathbf{x}, t)$ are treated as conventional divergence-free vector fields, $\mathbf{f}(\mathbf{x}, t)$ is the forcing that maintains the flow, p the total (kinetic plus magnetic) pressure, ν the kinematic viscosity, and η the magnetic diffusivity. The Fourier-space representation of equation (1) is given explicitly in appendix A (equations (5a) and (5b)).

The first question is of course how do we construct this d -dimensional closure model for MHD turbulence? Theoretically, the full MHD equations suffer from the same closure problems—and hence analytical progress—as the Navier–Stokes equation for fluid turbulence [33]. We recall that in fluid turbulence, theoretical progress in understanding the two-point correlation function stems first from a Quasi-Normal approximation which allows rewriting fourth-order moments as sums of products of different second-order moments. Then, the successive use of an (phenomenological) eddy-damping rate and *Markovianization* leads to a closed equation for the fluid kinetic energy spectrum in the EDQNM model. We follow a similar approach, beginning with the incompressible MHD equations, to derive the corresponding equations for the fluid $E^u(k)$ and magnetic $E^b(k)$ energy spectra (see appendices A–C for detailed derivations). Up to the definition of scalar spectral energies (see equation (15) in appendix A), the formulation proceeds in three dimensions; dimensionality then enters only through the geometric structure of the triadic interaction tensors $\mathcal{P}^{(k)}$ and $\mathcal{C}^{(k)}$ respectively. The extension to arbitrary (including non-integer) dimensions is therefore treated as a continuation of these scalar energetics.

$$\frac{d}{dt} E^u(k, t) = T_{(s)}^u(k) + T_{(c)}^u(k) - 2\nu k^2 E^u(k, t); \quad (2a)$$

$$\frac{d}{dt} E^b(k, t) = T_{(s)}^b(k) + T_{(c)}^b(k) - 2\eta k^2 E^b(k, t). \quad (2b)$$

The transfer terms are conveniently expressed in a form which underlines the distinct contributions from the self [subscript (s)] and coupled [subscript (c)] terms:

$$T_{(s)}^u(k) = 8K_d \int_{\Delta_k} d\mathbf{q} \, d\mathbf{p} \, W_d(\Delta_k) \theta_{kpq}^u \frac{k}{pq} b_{kpq} [k^{d-1} E^u(p) - p^{d-1} E^u(k)] E^u(q); \quad (3a)$$

$$T_{(c)}^u(k) = 8K_d \int_{\Delta_k} dq dp W_d(\Delta_k) \theta_{kpq}^b \frac{k}{pq} c_{kpq} [k^{d-1} E^b(p) - p^{d-1} E^u(k)] E^b(q); \quad (3b)$$

$$T_{(s)}^b(k) = 8K_d \int_{\Delta_k} dq dp W_d(\Delta_k) \theta_{qkp}^b \frac{k}{pq} h_{kpq} [k^{d-1} E^b(p) - p^{d-1} E^b(k)] E^u(q); \quad (3c)$$

$$T_{(c)}^b(k) = 8K_d \int_{\Delta_k} dq dp W_d(\Delta_k) \theta_{pqk}^b \frac{p}{kq} c_{pqk} [k^{d-1} E^u(p) - p^{d-1} E^b(k)] E^b(q). \quad (3d)$$

In equations (2) and (3), and what follows, k , p , and q are wavenumbers and the superscripts (u) and (b) always denote the fluid and magnetic fields, respectively. The integrals are over triads Δ_k formed from triangles with sides $\mathbf{k}$, $\mathbf{p}$, $\mathbf{q}$, and the time-scales θ_{kpq}^u and θ_{kpq}^b are a consequence of the eddy-damping and Markovian assumption. Furthermore, the explicit role of dimensions, which arise from the geometry of these triads in d -dimensional space, lead to an explicitly dimensional prefactor K_d , the weight of different triadic contributions $W_d(\Delta_k)$, the coupling coefficients b_{kpq} , c_{kpq} , and h_{kpq} (see appendix A).

The basic phenomenology of the primitive MHD equations are already apparent in the structure of our closure model. The self-interaction terms ensures the transfer of energy from different wavenumbers while $\int_0^\infty dk T_{(s)}^u(k) = \int_0^\infty dk T_{(s)}^b(k) = 0$ ensuring the conservation of energy. Further, the cross or coupling terms mediate the transfer of energy between the fluid and magnetic fields and, again for reasons of energy conservation, obey $\int_0^\infty dk (T_{(c)}^u + T_{(c)}^b) = 0$. Finally, it is easy to check that, for zero magnetic field $E^b = 0$, our model reduces to the d -dimensional fluid EDQNM equations [31]; similarly for $E^b \neq 0$ and choosing $d = 2$ or $d = 3$, we recover the two or three-dimensional EDQNM model, respectively, for MHD turbulence [34–36].

Trivially the dynamo question hinges on whether or not the total magnetic energy $\mathcal{E}^b = \int_0^\infty dk E^b(k)$ grows in time and eventually saturates to a nonzero value in the nonlinear regime. Starting from the evolution equations, it is easy to show that for an initial ($t = 0$) seed magnetic field such that the initial energies follow $\mathcal{E}_0^u \gg \mathcal{E}_0^b$ (allowing for terms quadratic in the magnetic energy to be omitted)

$$\frac{d\mathcal{E}^b}{dt} \approx \int_0^\infty dk (8K_d \lambda(k) - 2\eta k^2) E^b(k) \quad (4)$$

with $\lambda_d(k) = \int_{\Delta_k} dq dp W_d(\Delta_k) \theta_{pqk}^b \frac{p}{kq} c_{pqk} q^{d-1} E^u(p)$ depending only on the properties of the fluid and the dimension d . Equation (4) shows that the time behaviour of $\mathcal{E}^b$ is the result of two opposing effects, namely magnetic diffusion and the amplification by the velocity field, and it depends crucially on how kinetic energy is distributed across the Fourier modes of the velocity.

For ideal ($\nu = \eta = 0$) MHD, and with a finite number of modes (intrinsic to the MHD-EDQNM model), further progress is possible theoretically. This is because for $d > 2$, a global equipartition emerges as a thermal fixed point of the model. This ensures that for all dimensions $d > 2$, the seed magnetic field grows to asymptotically (in time) reach a state with $E^u(k) = E^b(k) \sim k^{d-1}$. However, $d = 2$ is special. Here, the conservation of magnetic potential ($\sum_k E^b(k)/k^2$), an additional constraint on these modes ensures a lack of global equipartition and thence $\mathcal{E}^u \gg \mathcal{E}^b$ for all times as long as $\mathcal{E}_0^u \gg \mathcal{E}_0^b$. Thus, in the ideal and finite-dimensional model, dynamo action is strictly possible for all dimensions $d > 2$.

But what happens for real flows which are dissipative and out-of-equilibrium? Here things are much harder to assess theoretically and we resort to evidence from numerical simulations to guide our intuition. We perform detailed numerical simulation of our MHD-EDQNM model (equations (2) and (3)) in dimensions $2 \leq d \leq 20$ with $0.01 \leq Pm \leq 5$ and $200 \leq Rm \leq 2 \times 10^4$. We refer the reader to appendix D for further details on the numerical method, parameters and tests on numerical convergence of our scheme.

We set up the numerical study of the dynamo problem in the following fashion. We first develop a statistically stationary state for the fluid kinetic spectrum by keeping the magnetic field switched off and driving the kinetic energy spectrum through a forcing spectrum (see appendix D) concentrated at large scales. This injection of energy is balanced with the net viscous dissipation rate $\epsilon^u = \int_0^\infty dk 2\nu k^2 E^u(k)$ to ensure a constant net kinetic energy $\mathcal{E}^u \equiv \int_0^\infty dk E^u(k) = 1.0$.

The steady state is characterized by a Kolmogorov spectrum $E^u(k) \sim k^{-5/3}$ for $d \gtrsim 3$ (albeit with an ever-pronounced bottleneck effect as d increases [31]) or a k^{-3} spectrum (due to the inverse cascade) for $d \approx 2$ cases [21, 28, 37]. The magnetic spectrum interaction is switched on with a initial seed of magnetic energy with $\mathcal{E}_0^b = 10^{-12}$. (We have also performed simulations with the initial seed as large as $\mathcal{E}_0^b = 10^{-2}$ and found our results robust to the choice of the seed magnetic field.) With the interaction on, the forcing is adjusted slightly to match the net dissipation rate which now includes the magnetic

dissipation rate ϵ^b . In what follows, the time when the magnetic field is switched on is set as $t = 0$, and the dynamo problem is studied at subsequent times.

3. The dynamo-no-dynamo phase diagram

To study the dynamo effect, we find it useful to define the measure $R(t) = \frac{\mathcal{E}^b(t)}{\mathcal{E}^u(t)}$ and observe its temporal behaviour for different dimensions. In figure 1, we show representative semilog plots of $R(t)$ versus time for several different dimensions. For two dimensional flows and as expected [38, 39], we have no dynamo action as $R(t)$, after an initial growth, decays in time. This transient growth followed by decay in two dimensions is consistent with earlier results [40] and reflects the suppression of sustained dynamo action, where field-line stretching is insufficient to overcome diffusive losses.

The three-dimensional case is just as clear: $R(t)$ increases, exponentially as evident from the linear rise on a semilog scale, and eventually saturates to value slightly larger than 0.5 indicating dynamo action. What is interesting is the behaviour for other dimensions. Clearly, there seems to be dimensions $d \geq 2.0$ as well as dimensions much larger than $d = 3.0$ where the dynamo fails. In fact in higher dimensions we do see an initial rise in $R(t)$ that becomes unsustainable with time. All of this suggests that at least within the MHD-EDQNM phenomenology there must exist a lower critical dimension $d_L \geq 2.0$ and a finite upper critical dimension $d_U \gg 3.0$ which dictates the dynamo—no-dynamo phase boundary. Furthermore, within the dynamo phase $d_L \leq d \leq d_U$, the growth of the magnetic energy is clearly exponential — $\exp\{\lambda t\}$ — before signs of saturation stemming from the nonlinearity. In the inset of figure 1 we show a plot of this normalized (with the viscous time-scale) growth rate $\tilde{\lambda}$ vs d (within the dynamo phase; see figure 3(b)) and find maximal growth for $d = 3$ for three different values of Pm.

Is it possible to have a theoretical explanation, starting from the equations of motion, which suggests such a phase diagram? While the short answer is, unfortunately, no, a scrutiny of the MHD-EDQNM model suggests that in the coupled set of equations, dynamo action for $d_L \leq d \leq d_U$ can only be a consequence of a predominant energy transfer from $\mathbf{u} \rightarrow \mathbf{b}$, with the transfer term acting as an effective forcing on the magnetic field. This preferential transfer of energy (at scales larger than those where the diffusive damping becomes strong) leads to an increasing $R(t)$ followed by an eventual saturation stemming from the nonlinearity (negligible at short times) and damping. Similarly, for $d < d_L$ or $d > d_U$ the large-scale energy transfer ought to be, preferentially, from $\mathbf{b} \rightarrow \mathbf{u}$, even if there is a net $\mathbf{u} \rightarrow \mathbf{b}$ transfer at smaller scales. This is because at small scales the magnetic dissipation term acts as a counter to the net *pumping* from the fluid field.

The argument outlined above is admittedly heuristic and a consequence of what we see in figure 1. The only way to make this argument plausible is to numerically analyse the spectral properties of the interaction terms in equation (3). In figure 2 we plot, scale-by-scale, $T_{(s)}^b(k) + T_{(c)}^b(k)$ together with the magnetic diffusion term $2\eta k^2 E^b$ at (inset) short ($t = 1$) and long ($t = 30$) times for (a) $d = 2.03$, (b) $d = 4.0$ and (c) $d = 8.0$. Furthermore, we calculate and show the net transfer $\Phi(k) = T_{(s)}^b(k) + T_{(c)}^b(k) - 2\eta k^2 E^b$ which is a clear indication of the scales which leak ($\Phi(k) < 0$) or pile on ($\Phi(k) > 0$) magnetic energy. However, as equation (4) suggest, the dynamo action is essentially an outcome of the integral of $\Phi(k)$; to make this point succinct, we also show in the same figure the cumulative integral $\int_0^k \Phi(k)$ as a function of the wavenumber k . Clearly, as $k \rightarrow \infty$, this $\int_0^k \Phi(k) > 0$ for $d_L \leq d \leq d_U$ and $\int_0^k \Phi(k) < 0$ for $d < d_L$ or $d > d_U$.

Figure 2 is then a clear illustration of our conjecture and consistent with observations in figure 1. At short times, the cumulative transfer (solid black line) is strictly *forcing* leading to an initial growth of the total magnetic energy. At long times, however, the situation is more delicate as the final state depends on the interaction of the fluid and magnetic components. Note that unlike kinematic models where the fluid component (velocity field or kinetic energy spectrum) is frozen, our MHD-EDQNM is able to go beyond the linear regime and provide a definitive answer to the dynamo problem. The final steady state of our MHD-EDQNM systems strongly depends on the dimension. For $d = 4.0$ the net transfer is strictly positive leading to dynamo action as seen in figure 1. A close inspection of the net transfer $\Phi(k)$ and its cumulative integral underlines this effect strongly. Furthermore, the small scales of pumping allow for a lack of compensation from the diffusive term leading to growth of the magnetic energy. Indeed, for such dimensions $d_L \leq d \leq d_U$, we see (figure 3(a)) that at long times there is a scale-by-scale cancellation of the pumping and damping leading to the saturation of magnetic energies and dynamo action.

However, for dimensions $d = 2.03$ and $d = 8.0$ which are clearly in the no-dynamo phase (figure 1), the spectral properties are more involved. At low wavenumbers (with negligible damping), the net

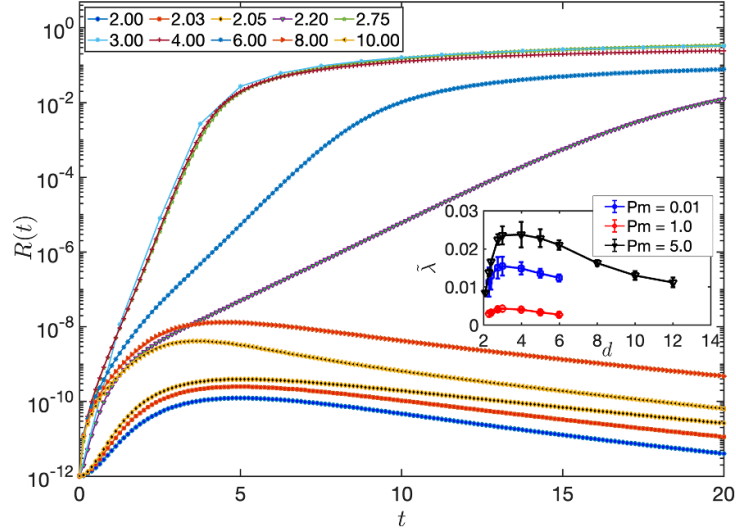


Figure 1. A semilog plot of $R(t)$ vs t for several different dimensions. For dimensions $d \leq d_L$ and $d \geq d_U$, the magnetic energy, after an initial increase, decreases with time indicating no sustained dynamo action. For dimensions $d_L \leq d \leq d_U$, the magnetic energy increases exponentially in time with an eventual dimension-dependent saturation. Inset: A plot of the normalized (with the viscous time-scale) growth rate $\tilde{\lambda}$ vs d for $d_L \leq d \leq d_U$ and different Prandtl numbers (see legend). Clearly, $d = 3$ corresponds to the fastest rate of dynamo growth.

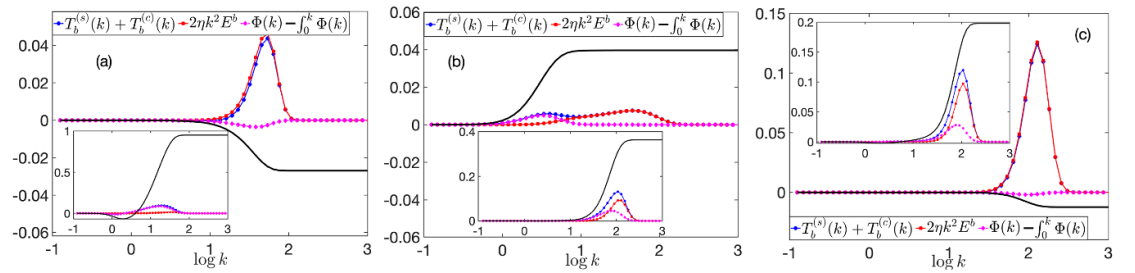


Figure 2. Scale-by-scale plots of $T_b^{(s)}(k) + T_b^{(c)}(k)$, the effective magnetic diffusive term $2\eta k^2 E^b$, the difference $\Phi(k) \equiv T_b^{(s)}(k) + T_b^{(c)}(k) - 2\eta k^2 E^b$, and the cumulative sum of the difference $\int_0^k \Phi(k)$, all of which are normalized by the magnetic energy at the same time instant, for (a) $d_L \geq d = 2.03$ (b) $d_L \leq d = 4.0 \leq d_U$ and (c) $d_U \leq d = 8.0$ at short (inset) $t = 1.0$ and long (close-to-saturation) $t = 20$ times. The scale-by-scale behaviour of $\Phi(k)$ is a clear measure of the effective pumping or dissipation of magnetic energy and the cumulative sum shows the net effect of the combined action of the transfer terms and magnetic dissipation. The insets also underline why at short times there is always an increase in the magnetic energy; at longer times, depending on the dimension, there is a net decrease or increase of the same leading to a dimension-dependent dynamo-no-dynamo phase diagram.

transfer is mainly from $\mathbf{b} \rightarrow \mathbf{u}$ leading to a depletion of magnetic energy. While there is still a persistent net $\mathbf{u} \rightarrow \mathbf{b}$ transfer for such dimensions, these happen at large wavenumbers (unlike what is seen for $d = 4.0$) and hence damped out by the magnetic diffusivity. Such a spectral analysis thus is useful in providing not a theory, but an understanding of where the dynamo–no-dynamo transition may happen as a function of the dimension d . In particular, and as already suggested in figure 1, it clearly shows the possibility of a lower d_L and upper d_U critical dimension, tied to the diffusive scales, marking out the boundary between dynamo and a no-dynamo phase.

All of this now leads us to construct the phase diagram for the dynamo–no-dynamo transition. In figure 3(b) we show a *space-time* pseudo-colour plot of $R(t)$ as a function of dimensions and time. Clearly at long times, $R(t) \rightarrow 0$ for $d \leq 2.15 \approx d_L$ and for $d \geq 7.5 \approx d_U$. These dimensions are indicated by the vertical broken lines in the figure and our numerical simulations of the MHD-EDQNM model predict dynamo action for all dimensions which lie in between these two. In the absence of a theory, the precise value of these dimensions are moot; indeed, as discussed in appendix E, d_L and d_U depend on Pm and Rm . What remains a challenge is to find a more complete theory of why such dimensions exist in the first place as well as a systematic study of the dependence of d_L and d_U on Pm and Rm .

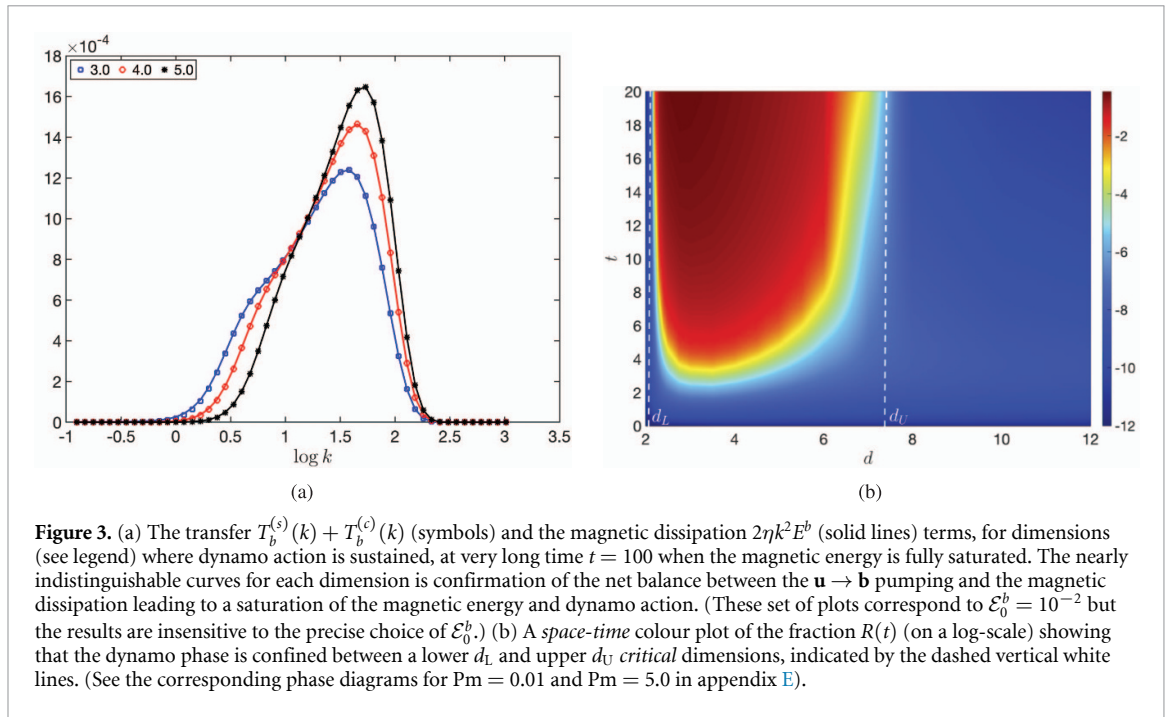


Figure 3. (a) The transfer $T_b^{(s)}(k) + T_b^{(c)}(k)$ (symbols) and the magnetic dissipation $2\eta k^2 E^b$ (solid lines) terms, for dimensions (see legend) where dynamo action is sustained, at very long time $t = 100$ when the magnetic energy is fully saturated. The nearly indistinguishable curves for each dimension is confirmation of the net balance between the $\mathbf{u} \rightarrow \mathbf{b}$ pumping and the magnetic dissipation leading to a saturation of the magnetic energy and dynamo action. (These set of plots correspond to $\mathcal{E}_0^b = 10^{-2}$ but the results are insensitive to the precise choice of $\mathcal{E}_0^b$.) (b) A space-time colour plot of the fraction $R(t)$ (on a log-scale) showing that the dynamo phase is confined between a lower d_L and upper d_U critical dimensions, indicated by the dashed vertical white lines. (See the corresponding phase diagrams for $Pm = 0.01$ and $Pm = 5.0$ in appendix E).

4. Conclusions

Why are these results important? While d_L and d_U must depend on Rm and Pm (as shown in appendix E), a thorough, systematic study of this, while necessary for the future, is unlikely to reveal new physical phenomena. Rather, we believe that the interesting point is the very existence of a range of dimensions outside which the dynamo effect is not possible. This is not an obvious phenomenon. So far, it had been observed only in the Kazantsev model [19], which, however, has very special temporal correlations. Our results, therefore, provide strong evidence that the existence of critical dimensions is not specific to the Kazantsev model and may be a general feature of d -dimensional MHD. Hence, theoretical approaches such as ours, provide a fresh perspective, in the spirit of critical phenomena, by establishing the existence of lower and upper critical dimensions. These results, in isolating the vital role played by triadic interactions, provides a strong framework for understanding the dynamo problem rooted in the full nonlinear set of MHD equation. Hence we complement several d -dimensional studies of hydrodynamic turbulence which have shed significant light on understanding critical questions of intermittency [21–27] in classical fluid turbulence. In general, the success in charting the dynamo phase diagram suggests that analogous spectral closure parametrizations hold strong promise for a other complex settings such as inhomogeneous turbulence [41, 42] and geophysical flows [43]. Finally, our d -dimensional (non linear) closure model is adapted easily to include effects of compressibility [19], non-zero helicity and cross-helicity [44–46], spatial regularity of the velocity field [47] and study the dynamo problem over a wide range of Pm and Rm inaccessible to DNSs as well as overcoming the limitations of previous theoretical approaches.

Acknowledgements

S D M, S S R and D V thank the Indo-French Centre for Applied Mathematics (IFCAM) for financial support. S.S.R and D.V. also acknowledge the CEFIPRA Project No 6704-1 for support. The simulations were performed on the ICTS clusters *Tetris* and *Contra*. SSR acknowledges SERB-DST (India) projects STR/2021/000023 and CRG/2021/002766 for financial support and would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme *Anti-diffusive dynamics: from sub-cellular to astrophysical scales* (EPSRC EP/R014604/1), where part of the work on this paper was undertaken. This research was supported in part by the International Centre for Theoretical Sciences (ICTS) for participating in the programs—*Field Theory and Turbulence* (code:ICTS/ftt2023/12) and *Turbulence: Problems at the Interface of Mathematics and Physics* (code: ICTS/TPIMP2020/12). SSR acknowledges the support of the DAE, Govt. of India, under project no. 12-R&D-TFR-5.10-1100 and project no. RTI4001.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. The quasi-normal approximation

The derivation of the closure model follows best from the Fourier space representation of the MHD equations, expressed conveniently in a symmetric form between the fluid and magnetic fields, written in component form with Greek indices:

$$\left[\frac{d}{dt} + \nu k^2 \right] u_\alpha(\mathbf{k}, t) = \mathcal{P}_{\alpha\rho\gamma}^{(k)} \int d\mathbf{q} \int d\mathbf{p} [u_\rho(\mathbf{q})u_\gamma(\mathbf{p}) - b_\rho(\mathbf{q})b_\gamma(\mathbf{p})] \delta(\mathbf{p} + \mathbf{q} - \mathbf{k}); \quad (5a)$$

$$\left[\frac{d}{dt} + \eta k^2 \right] b_\alpha(\mathbf{k}, t) = \mathcal{C}_{\alpha\rho\gamma}^{(k)} \int d\mathbf{q} \int d\mathbf{p} [b_\rho(\mathbf{q})u_\gamma(\mathbf{p}) + b_\rho(\mathbf{p})u_\gamma(\mathbf{q})] \delta(\mathbf{p} + \mathbf{q} - \mathbf{k}). \quad (5b)$$

The forcing term in the spectral equation for the velocity field, equation (5a), is omitted from the equations that follow but is implicitly assumed to be present. Its precise form is irrelevant to the derivation; it is only assumed to act at large scales, near the injection wavenumber $k \sim k_f$. The specific form of forcing adopted in the numerical implementation of the EDQNM-MHD equations is introduced and discussed later in appendix D. By defining $\mathbb{P}_{\alpha\beta}^{(k)} = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}$, we obtain

$$\mathcal{P}_{\alpha\rho\gamma}^{(k)} = -\frac{\ell}{2} \left(\mathbb{P}_{\alpha\rho}^{(k)} k_\gamma + \mathbb{P}_{\alpha\gamma}^{(k)} k_\rho \right); \quad (6a)$$

$$\mathcal{C}_{\alpha\rho\gamma}^{(k)} = -\frac{\ell}{2} (\delta_{\alpha\rho} k_\gamma - \delta_{\alpha\gamma} k_\rho) \quad (6b)$$

for the triadic interaction tensors, respectively.

The form of the generalized n th order spectral moment (for fields X^i)

$$\mathcal{S}_{\alpha_1\alpha_2\cdots\alpha_n}^{X^{(1)}X^{(2)}\cdots X^{(n)}}(k_1, k_2, \dots, k_{n-1}) = \left\langle X_{\alpha_1}^{(1)}(k_1) X_{\alpha_2}^{(2)}(k_2) \cdots X_{\alpha_{n-1}}^{(n-1)}(k_{n-1}) X_{\alpha_n}^{(n)} \left(-\sum_{i=1}^{n-1} k_i \right) \right\rangle \quad (7)$$

allows us to obtain the evolution equations for the second moments $\mathcal{S}^{uu}$ and $\mathcal{S}^{bb}$:

$$\left[\frac{d}{dt} + 2\nu k^2 \right] \mathcal{S}_{\alpha\beta}^{uu}(k) = \int_{\Delta_k} \mathcal{P}_{\beta\rho\gamma}^{(-k)} [\mathcal{S}_{\alpha\rho\gamma}^{uuu}(k, -q) - \mathcal{S}_{\alpha\rho\gamma}^{ubb}(k, -q)] + \text{c.c.} \Big|_{\alpha \leftrightarrow \beta}; \quad (8a)$$

$$\left[\frac{d}{dt} + 2\eta k^2 \right] \mathcal{S}_{\alpha\beta}^{bb}(k) = \int_{\Delta_k} \mathcal{C}_{\beta\rho\gamma}^{(-k)} [\mathcal{S}_{\alpha\rho\gamma}^{bbu}(k, -q) + \mathcal{S}_{\alpha\rho\gamma}^{bbu}(k, -p)] + \text{c.c.} \Big|_{\alpha \leftrightarrow \beta}. \quad (8b)$$

Here $\int_{\Delta_k} \equiv \int d\mathbf{q} \int d\mathbf{p} \delta(\mathbf{q} + \mathbf{p} - \mathbf{k})$, c.c. denotes complex conjugate, and $\alpha \leftrightarrow \beta$ implies the exchange of indices.

Similarly, the evolution of the third-order moments $\mathcal{S}^{uuu}$, $\mathcal{S}^{ubb}$, and $\mathcal{S}^{bbu}$ follows:

$$\begin{aligned} \left[\frac{d}{dt} + \nu(k^2 + q^2 + p^2) \right] \mathcal{S}_{\alpha\rho\gamma}^{uuu}(k, -q) &= \int d\mathbf{r} \mathcal{P}_{\alpha\mu\sigma}^{(k)} [\mathcal{S}_{\rho\gamma\mu\sigma}^{uuuu}(-q, -p, r) - \mathcal{S}_{\rho\gamma\mu\sigma}^{uubb}(-q, -p, r)] \\ &\quad + \int d\mathbf{r} \mathcal{P}_{\rho\mu\sigma}^{(-q)} [\mathcal{S}_{\alpha\gamma\mu\sigma}^{uuuu}(k, -p, r) - \mathcal{S}_{\alpha\gamma\mu\sigma}^{uubb}(k, -p, r)] \\ &\quad + \int d\mathbf{r} \mathcal{P}_{\gamma\mu\sigma}^{(-p)} [\mathcal{S}_{\alpha\rho\mu\sigma}^{uuuu}(k, -q, r) - \mathcal{S}_{\alpha\rho\mu\sigma}^{uubb}(k, -q, r)] \end{aligned} \quad (9a)$$

$$\begin{aligned} \left[\frac{d}{dt} + \nu k^2 + \eta(q^2 + p^2) \right] \mathcal{S}_{\alpha\rho\gamma}^{ubb}(k, -q) &= \int d\mathbf{r} \mathcal{P}_{\alpha\mu\sigma}^{(k)} [\mathcal{S}_{\rho\gamma\mu\sigma}^{bbuu}(-q, -p, r) - \mathcal{S}_{\rho\gamma\mu\sigma}^{bbbb}(-q, -p, r)] \\ &\quad + 2 \int d\mathbf{r} [\mathcal{C}_{\rho\mu\sigma}^{(-q)} \mathcal{S}_{\gamma\mu\alpha\sigma}^{bbuu}(-p, r, k) + \mathcal{C}_{\gamma\mu\sigma}^{(-p)} \mathcal{S}_{\rho\mu\alpha\sigma}^{bbuu}(-q, r, k)] \end{aligned} \quad (9b)$$

$$\left[\frac{d}{dt} + \nu p^2 + \eta(k^2 + q^2) \right] \mathcal{S}_{\alpha\rho\gamma}^{bbu}(k, -q) = \int d\mathbf{r} \mathcal{P}_{\gamma\mu\sigma}^{(-p)} [\mathcal{S}_{\alpha\rho\mu\sigma}^{bbuu}(k, -q, r) - \mathcal{S}_{\alpha\rho\mu\sigma}^{bbbb}(k, -q, r)] \\ + 2 \int d\mathbf{r} \left[\mathcal{C}_{\alpha\mu\sigma}^{(k)} \mathcal{S}_{\rho\mu\gamma\sigma}^{bbuu}(-q, r, -p) + \mathcal{C}_{\rho\mu\sigma}^{(-q)} \mathcal{S}_{\alpha\mu\gamma\sigma}^{bbuu}(k, r, -p) \right] \quad (9c)$$

$$\left[\frac{d}{dt} + \nu q^2 + \eta(k^2 + p^2) \right] \mathcal{S}_{\alpha\rho\gamma}^{bbu}(k, -p) = \int d\mathbf{r} \mathcal{P}_{\gamma\mu\sigma}^{(-q)} [\mathcal{S}_{\alpha\rho\mu\sigma}^{bbuu}(k, -p, r) - \mathcal{S}_{\alpha\rho\mu\sigma}^{bbbb}(k, -p, r)] \\ + 2 \int d\mathbf{r} \left[\mathcal{C}_{\alpha\mu\sigma}^{(k)} \mathcal{S}_{\rho\mu\gamma\sigma}^{bbuu}(-p, r, -q) + \mathcal{C}_{\rho\mu\sigma}^{(-p)} \mathcal{S}_{\alpha\mu\gamma\sigma}^{bbuu}(k, r, -q) \right]. \quad (9d)$$

A comparison between equations (8) and (9) underlines the closure problem inherent in such models: Solving for the n th moment is contingent on knowing the $(n+1)$ th moment. Hence suitable approximations are needed to close this hierarchy and find, for our problem, a closed form representation of the second-order moments. One such approach is the quasi-normal approximation which assumes that the statistics to be essentially Gaussian (with a vanishing cumulant) and hence

$$\mathcal{S}_{\alpha\beta\rho\gamma}^{X^{(1)}X^{(2)}X^{(3)}X^{(4)}}(k_1, k_2, k_3) = \mathcal{S}_{\alpha\beta}^{X^{(1)}X^{(2)}}(k_1) \mathcal{S}_{\rho\gamma}^{X^{(3)}X^{(4)}}(k_3) \delta(k_1 + k_2) + \mathcal{S}_{\alpha\rho}^{X^{(1)}X^{(3)}}(k_1) \mathcal{S}_{\beta\gamma}^{X^{(2)}X^{(4)}}(k_2) \delta(k_1 + k_3) \\ + \mathcal{S}_{\alpha\gamma}^{X^{(1)}X^{(4)}}(k_1) \mathcal{S}_{\beta\rho}^{X^{(2)}X^{(3)}}(k_2) \delta(k_2 + k_3). \quad (10)$$

This form allows us (with the further assumption $\langle u_\alpha(k_1) b_\beta(k_2) \rangle = 0$) to reduce equation (9) to

$$\left[\frac{d}{dt} + \nu(k^2 + p^2 + q^2) \right] \mathcal{S}_{\alpha\rho\gamma}^{uuu}(k, -q) = 2 \left[\mathcal{P}_{\alpha\mu\sigma}^{(k)} \mathcal{S}_{\rho\mu}^{uu}(-q) \mathcal{S}_{\gamma\sigma}^{uu}(-p) + \mathcal{P}_{\rho\mu\sigma}^{(-q)} \mathcal{S}_{\alpha\sigma}^{uu}(k) \mathcal{S}_{\gamma\mu}^{uu}(-p) \right. \\ \left. + \mathcal{P}_{\gamma\mu\sigma}^{(-p)} \mathcal{S}_{\alpha\mu}^{uu}(k) \mathcal{S}_{\rho\sigma}^{uu}(-q) \right] \quad (11a)$$

$$\left[\frac{d}{dt} + \nu k^2 + \eta(q^2 + p^2) \right] \mathcal{S}_{\alpha\rho\gamma}^{ubb}(k, -q) = 2 \left[\mathcal{C}_{\rho\mu\sigma}^{(-q)} \mathcal{S}_{\gamma\mu}^{bb}(-p) \mathcal{S}_{\alpha\sigma}^{uu}(k) + \mathcal{C}_{\gamma\mu\sigma}^{(-p)} \mathcal{S}_{\rho\mu}^{bb}(-q) \mathcal{S}_{\alpha\sigma}^{uu}(k) \right. \\ \left. - \mathcal{P}_{\alpha\mu\sigma}^{(k)} \mathcal{S}_{\rho\mu}^{bb}(-q) \mathcal{S}_{\gamma\sigma}^{bb}(-p) \right] \quad (11b)$$

$$\left[\frac{d}{dt} + \nu p^2 + \eta(k^2 + q^2) \right] \mathcal{S}_{\alpha\rho\gamma}^{bbu}(k, -q) = 2 \left[\mathcal{C}_{\alpha\mu\sigma}^{(k)} \mathcal{S}_{\rho\mu}^{bb}(-q) \mathcal{S}_{\gamma\sigma}^{uu}(-p) + \mathcal{C}_{\rho\mu\sigma}^{(-q)} \mathcal{S}_{\alpha\mu}^{bb}(k) \mathcal{S}_{\gamma\sigma}^{uu}(-p) \right. \\ \left. - \mathcal{P}_{\gamma\mu\sigma}^{(-p)} \mathcal{S}_{\alpha\mu}^{bb}(k) \mathcal{S}_{\rho\sigma}^{bb}(-q) \right] \quad (11c)$$

$$\left[\frac{d}{dt} + \nu q^2 + \eta(k^2 + p^2) \right] \mathcal{S}_{\alpha\rho\gamma}^{bbu}(k, -p) = 2 \left[\mathcal{C}_{\alpha\mu\sigma}^{(k)} \mathcal{S}_{\rho\mu}^{bb}(-p) \mathcal{S}_{\gamma\sigma}^{uu}(-q) + \mathcal{C}_{\rho\mu\sigma}^{(-p)} \mathcal{S}_{\alpha\mu}^{bb}(k) \mathcal{S}_{\gamma\sigma}^{uu}(-q) \right. \\ \left. - \mathcal{P}_{\gamma\mu\sigma}^{(-q)} \mathcal{S}_{\alpha\mu}^{bb}(k) \mathcal{S}_{\rho\sigma}^{bb}(-p) \right]. \quad (11d)$$

This form allows us, by defining

$$\left[\frac{d}{dt} + \omega \right]^{-1} \cdot = \hat{\theta}(\omega) \cdot = \int_0^t ds \cdot e^{-(t-s)\omega}, \quad (12)$$

to invert equation (11) and, on substitution in equation (8), obtain

$$\left[\frac{d}{dt} + 2\nu k^2 \right] \mathcal{S}_{\alpha\beta}^{uu}(k) = \int_{\Delta_k} 2\hat{\theta}(\omega_{kpq}^u) \mathcal{P}_{\beta\rho\gamma}^{(-k)} \left[\mathcal{P}_{\alpha\mu\sigma}^{(k)} \mathcal{S}_{\rho\mu}^{uu}(-q) \mathcal{S}_{\gamma\sigma}^{uu}(-p) + \mathcal{P}_{\rho\mu\sigma}^{(-q)} \mathcal{S}_{\alpha\sigma}^{uu}(k) \mathcal{S}_{\gamma\mu}^{uu}(-p) \right. \\ \left. + \mathcal{P}_{\gamma\mu\sigma}^{(-p)} \mathcal{S}_{\alpha\mu}^{uu}(k) \mathcal{S}_{\rho\sigma}^{uu}(-q) \right] + \text{c.c.} \Big|_{\alpha \leftrightarrow \beta} \\ - \int_{\Delta_k} 2\hat{\theta}(\omega_{kpq}^b) \mathcal{P}_{\beta\rho\gamma}^{(-k)} \left[\mathcal{C}_{\rho\mu\sigma}^{(-q)} \mathcal{S}_{\alpha\sigma}^{uu}(k) \mathcal{S}_{\gamma\mu}^{bb}(-p) + \mathcal{C}_{\gamma\mu\sigma}^{(-p)} \mathcal{S}_{\alpha\sigma}^{uu}(k) \mathcal{S}_{\rho\mu}^{bb}(-q) \right. \\ \left. - \mathcal{P}_{\alpha\mu\sigma}^{(k)} \mathcal{S}_{\rho\mu}^{bb}(-q) \mathcal{S}_{\gamma\sigma}^{bb}(-p) \right] + \text{c.c.} \Big|_{\alpha \leftrightarrow \beta} \quad (13a) \\ \left[\frac{d}{dt} + 2\eta k^2 \right] \mathcal{S}_{\alpha\beta}^{bb}(k) = \int_{\Delta_k} 2\hat{\theta}(\omega_{pkq}^b) \mathcal{C}_{\beta\rho\gamma}^{(-k)} \left[\mathcal{C}_{\alpha\mu\sigma}^{(k)} \mathcal{S}_{\rho\mu}^{bb}(-q) \mathcal{S}_{\gamma\sigma}^{uu}(-p) + \mathcal{C}_{\rho\mu\sigma}^{(-q)} \mathcal{S}_{\alpha\mu}^{bb}(k) \mathcal{S}_{\gamma\sigma}^{uu}(-p) \right. \\ \left. - \mathcal{P}_{\gamma\mu\sigma}^{(-p)} \mathcal{S}_{\alpha\mu}^{bb}(k) \mathcal{S}_{\rho\sigma}^{bb}(-q) \right]$$

$$+ \int_{\Delta_k} 2\hat{\theta}(\omega_{kpq}^b) \mathcal{C}_{\beta\rho\gamma}^{(-k)} \left[\mathcal{C}_{\alpha\mu\sigma}^{(k)} \mathcal{S}_{\rho\mu}^{bb}(-p) \mathcal{S}_{\gamma\sigma}^{uu}(-q) + \mathcal{C}_{\rho\mu\sigma}^{(-p)} \mathcal{S}_{\alpha\mu}^{bb}(k) \mathcal{S}_{\gamma\sigma}^{uu}(-q) \right. \\ \left. - \mathcal{P}_{\gamma\mu\sigma}^{(-q)} \mathcal{S}_{\alpha\mu}^{bb}(k) \mathcal{S}_{\rho\sigma}^{bb}(-p) \right] + \text{c.c.} \Big|_{\alpha \leftrightarrow \beta}. \quad (13b)$$

The frequencies defined in the operators $\hat{\theta}$ are:

$$\omega_{kpq}^u = \omega_k^u + \omega_p^u + \omega_q^u, \quad \omega_k^u = \nu k^2, \\ \omega_{kpq}^b = \omega_k^u + \omega_p^b + \omega_q^b, \quad \omega_k^b = \eta k^2. \quad (14)$$

Isotropy helps to simplify this problem further. By writing the second moment in terms of the rotationally invariant second-rank tensors $\delta_{\alpha\beta}$ and $\hat{k}_\alpha, \hat{k}_\beta$:

$$\mathcal{S}_{\alpha\beta}^{XX}(k) = (c_1 \delta_{\alpha\beta} + c_2 \hat{k}_\alpha \hat{k}_\beta) \quad (15)$$

and since, by definition, $\mathcal{S}_{\alpha\alpha}^{XX}(k) = \mathcal{U}^X(k)$, we obtain for the incompressible problem $c_1 = \frac{1}{d-1} = -c_2$.

Incompressibility demands $\hat{k}_\alpha \mathcal{S}_{\alpha\beta}^{XX} = 0$, hence $c_1 + c_2 = 0$. Introducing the trace of the second moment as $\mathcal{S}_{\alpha\alpha}^{XX}(k) = \mathcal{U}^X(k)$, allows us to rewrite equation (15) as

$$\mathcal{S}_{\alpha\beta}^{XX}(k) = \frac{1}{(d-1)} \mathbb{P}_{\alpha\beta}^{(k)} \mathcal{U}^X(k). \quad (16)$$

Furthermore, the operators within the isotropic model obey:

$$\mathbb{P}_{\alpha\beta}^{(k)} \mathbb{P}_{\alpha\rho}^{(k)} = \mathbb{P}_{\beta\rho}^{(k)} \quad ; \quad \mathbb{P}_{\alpha\beta}^{(k)} \mathbb{P}_{\alpha\beta}^{(k)} = d-1 \\ \mathcal{P}_{\alpha\rho\gamma}^{(k)} = \mathcal{P}_{\alpha\gamma\rho}^{(k)} \quad ; \quad \mathcal{C}_{\alpha\rho\gamma}^{(k)} = -\mathcal{C}_{\alpha\gamma\rho}^{(k)} \\ \mathbb{P}_{\alpha\beta}^{(k)} \mathcal{P}_{\beta\rho\gamma}^{(k)} = \mathcal{P}_{\alpha\rho\gamma}^{(k)} \quad ; \quad \mathbb{P}_{\alpha\beta}^{(k)} \mathcal{C}_{\beta\rho\gamma}^{(k)} = \mathcal{C}_{\alpha\rho\gamma}^{(k)}. \quad (17)$$

By exploiting these symmetries, it is then a matter of algebra to show

$$\left[\frac{d}{dt} + 2\nu k^2 \right] \mathcal{U}^u(k) = \frac{4}{(d-1)^2} \int_{\Delta_k} \hat{\theta}(\omega_{kpq}^u) \mathcal{P}_{\alpha\rho\gamma}^{(k)} \left[-\mathcal{P}_{\alpha\mu\sigma}^{(k)} \mathbb{P}_{\rho\mu}^{(q)} \mathbb{P}_{\gamma\sigma}^{(p)} \mathcal{U}^u(q) \mathcal{U}^u(p) \right. \\ \left. + \mathcal{P}_{\rho\mu\sigma}^{(q)} \mathbb{P}_{\alpha\sigma}^{(k)} \mathbb{P}_{\gamma\mu}^{(p)} \mathcal{U}^u(k) \mathcal{U}^u(p) + \mathcal{P}_{\gamma\mu\sigma}^{(p)} \mathbb{P}_{\alpha\mu}^{(k)} \mathbb{P}_{\rho\sigma}^{(q)} \mathcal{U}^u(k) \mathcal{U}^u(q) \right] \\ - \frac{4}{(d-1)^2} \int_{\Delta_k} \hat{\theta}(\omega_{kpq}^b) \mathcal{P}_{\alpha\rho\gamma}^{(k)} \left[\mathcal{P}_{\alpha\mu\sigma}^{(k)} \mathbb{P}_{\rho\mu}^{(q)} \mathbb{P}_{\gamma\sigma}^{(p)} \mathcal{U}^b(q) \mathcal{U}^b(p) \right. \\ \left. + \mathcal{C}_{\rho\mu\sigma}^{(q)} \mathbb{P}_{\alpha\sigma}^{(k)} \mathbb{P}_{\gamma\mu}^{(p)} \mathcal{U}^u(k) \mathcal{U}^b(p) + \mathcal{C}_{\gamma\mu\sigma}^{(p)} \mathbb{P}_{\alpha\mu}^{(k)} \mathbb{P}_{\rho\sigma}^{(q)} \mathcal{U}^u(k) \mathcal{U}^b(q) \right] \quad (18a)$$

$$\left[\frac{d}{dt} + 2\eta k^2 \right] \mathcal{U}^b(k) = \frac{4}{(d-1)^2} \int_{\Delta_k} \hat{\theta}(\omega_{kpq}^b) \mathcal{C}_{\alpha\rho\gamma}^{(k)} \left[-\mathcal{P}_{\gamma\mu\sigma}^{(p)} \mathbb{P}_{\alpha\mu}^{(k)} \mathbb{P}_{\rho\sigma}^{(q)} \mathcal{U}^b(k) \mathcal{U}^b(q) \right. \\ \left. - \mathcal{C}_{\alpha\mu\sigma}^{(k)} \mathbb{P}_{\rho\mu}^{(q)} \mathbb{P}_{\gamma\sigma}^{(p)} \mathcal{U}^b(q) \mathcal{U}^u(p) + \mathcal{C}_{\rho\mu\sigma}^{(q)} \mathbb{P}_{\alpha\mu}^{(k)} \mathbb{P}_{\gamma\sigma}^{(p)} \mathcal{U}^b(k) \mathcal{U}^u(p) \right] \\ + \frac{4}{(d-1)^2} \int_{\Delta_k} \hat{\theta}(\omega_{kpq}^u) \mathcal{C}_{\alpha\rho\gamma}^{(k)} \left[-\mathcal{P}_{\gamma\mu\sigma}^{(q)} \mathbb{P}_{\alpha\mu}^{(k)} \mathbb{P}_{\rho\sigma}^{(p)} \mathcal{U}^b(k) \mathcal{U}^b(p) \right. \\ \left. - \mathcal{C}_{\alpha\mu\sigma}^{(k)} \mathbb{P}_{\rho\mu}^{(p)} \mathbb{P}_{\gamma\sigma}^{(q)} \mathcal{U}^b(p) \mathcal{U}^u(q) + \mathcal{C}_{\rho\mu\sigma}^{(p)} \mathbb{P}_{\alpha\mu}^{(k)} \mathbb{P}_{\gamma\sigma}^{(q)} \mathcal{U}^b(k) \mathcal{U}^u(q) \right]. \quad (18b)$$

It is useful to introduce geometric coefficients

$$a_{kqp} = a_{kpq} = -\frac{1}{k^2} \mathcal{P}_{\alpha\rho\gamma}^{(k)} \mathcal{P}_{\alpha\mu\sigma}^{(k)} \mathbb{P}_{\rho\mu}^{(q)} \mathbb{P}_{\gamma\sigma}^{(p)} \quad (19a)$$

$$b_{kpq} = -\frac{2}{k^2} \mathcal{P}_{\mu\rho\gamma}^{(k)} \mathcal{P}_{\gamma\mu\sigma}^{(p)} \mathbb{P}_{\rho\sigma}^{(q)} \quad (19b)$$

$$c_{kpq} = \frac{2}{k^2} \mathcal{P}_{\mu\rho\gamma}^{(k)} \mathcal{C}_{\gamma\mu\sigma}^{(p)} \mathbb{P}_{\rho\sigma}^{(q)} \quad (19c)$$

$$g_{kqp} = g_{kpq} = -\frac{2}{k^2} \mathcal{C}_{\alpha\rho\gamma}^{(k)} \mathcal{C}_{\alpha\mu\sigma}^{(k)} \mathbb{P}_{\rho\mu}^{(q)} \mathbb{P}_{\gamma\sigma}^{(p)} \quad (19d)$$

$$h_{kpq} = -\frac{2}{k^2} \mathcal{C}_{\mu\rho\gamma}^{(k)} \mathcal{C}_{\rho\mu\sigma}^{(p)} \mathbb{P}_{\gamma\sigma}^{(q)}. \quad (19e)$$

These geometric coefficients depend on the angle of the triangle formed by $\mathbf{k}, \mathbf{p}, \mathbf{q}$, and expressed in terms of the cosines of the resultant angles. Specifically, for a triangle formed by sides of length

k, p, q and defining the cosines of the angles opposite to their sides as x, y, z , the following relations hold [31, 34, 35]

$$x^2 + y^2 + z^2 = 1 - 2xyz$$

$$a_{kpq} = \frac{1}{2} \left(1 - 2y^2z^2 - xyz + \frac{(d-3)}{2} [2 - y^2 - z^2] \right) \quad (20a)$$

$$b_{kpq} = \frac{p}{k} \left(z^3 + xy + \frac{(d-3)}{2} [z + xy] \right) \quad (20b)$$

$$c_{kpq} = \frac{p}{k} \left(z[1 - y^2] + \frac{(d-3)}{2} [z + xy] \right) \quad (20c)$$

$$g_{kpq} = 1 + xyz + \frac{(d-3)}{2} [2 - y^2 - z^2] \quad (20d)$$

$$h_{kpq} = \frac{p}{2k} ((d-1)[z + xy]). \quad (20e)$$

Formally, the evolution equation for the spectral energies can be written in the form

$$\left[\frac{d}{dt} + 2\nu k^2 \right] \mathcal{U}^X = \int_{\Delta_k} \mathcal{T}^X(\Delta_k), \quad (21)$$

where $\mathcal{T}^X(\Delta_k)$ is the transfer integrand for the field X arising from a particular triad $\mathbf{k}, \mathbf{p}, \mathbf{q}$. The constructed integrand depends only on the geometry of the triad; this allows us to integrate out additional degrees of freedom in such integral $\int_{\Delta_k}$.

In d -dimensions such integrals can be simplified as follows. By construction, the transfer term is a function of just the magnitude and angle for a pair of wavevectors $\mathbf{k}$ and $\mathbf{q}$. In d dimensions, the Cartesian coordinates, radius, and spherical angles are related as

$$\begin{aligned} x_1 &= r \cos(\phi_1) \\ x_2 &= r \sin(\phi_1) \cos(\phi_2) \\ &\vdots \\ x_{n-1} &= r \sin(\phi_1) \cdots \sin(\phi_{n-2}) \cos(\phi_{n-1}) \\ x_n &= r \sin(\phi_1) \cdots \sin(\phi_{n-2}) \sin(\phi_{n-1}). \end{aligned} \quad (22)$$

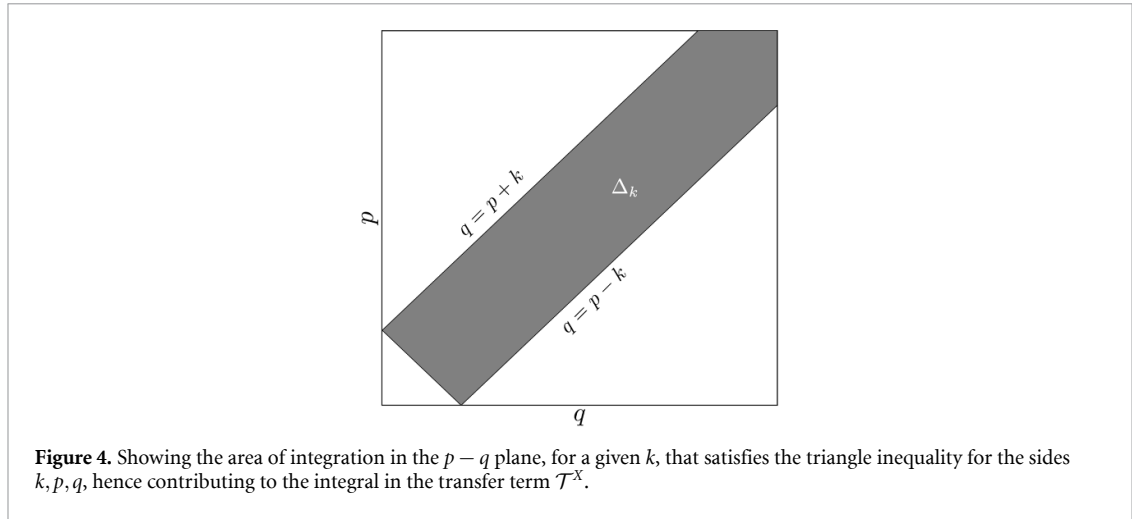
We align, for convenience, our axis such that $\mathbf{k}$ is along x_1 and denote $\phi_1 = \beta$. We now integrate out the remaining angles $\phi_2, \dots, \phi_{n-1}$ which form a $d-2$ dimensional sphere yielding

$$\int d^{d-1}\Omega = S_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}. \quad (23)$$

This now allows us to further evaluate the evolution equation for the spectral energies as:

$$\begin{aligned} \left[\frac{d}{dt} + 2\nu k^2 \right] \mathcal{U}^X &= \int_{\Delta_k} \mathcal{T}^X(\Delta_k) \\ &= \int d\mathbf{q} d\mathbf{p} \delta(\mathbf{p} + \mathbf{q} - \mathbf{k}) \mathcal{T}^X(\Delta_k) \\ &= \int d^d q \mathcal{T}^X(\Delta_k) = \int dq q^{d-1} d^{d-1}\Omega \mathcal{T}^X(\Delta_k) \\ &= \int_{q=0}^{\infty} dq q^{d-1} \int_{\beta=0}^{\pi} d\beta \sin^{d-2}(\beta) \mathcal{T}^X(\Delta_k) \int d^{d-2}\Omega \\ &= \int_{q=0}^{\infty} dq q^{d-1} \int_{y=-1}^{y=1} dy \sin^{d-3}(\beta) \mathcal{T}^X(\Delta_k) S_{d-2}. \end{aligned} \quad (24)$$

To further simplify equation (24), we use the sine law of triangle and following change of variables:



$$\begin{aligned} \frac{\sin(\alpha)}{k} &= \frac{\sin(\beta)}{p} \\ \int_0^\infty dq \int_0^1 dy &= \mathbf{J} \left[\frac{\partial(q(q,p), y(q,p))}{\partial(q,p)} \right] \int_0^\infty dq \int_{|k-q|}^{|k+q|} dp \\ &= \left(\frac{p}{kq} \right) \int_{\Delta_k} dq dp \end{aligned} \quad (25)$$

where $\mathbf{J} \left[\frac{\partial(q(q,p), y(q,p))}{\partial(q,p)} \right]$ is the Jacobian for change of variables, and $\int_{\Delta_k} dq dp = \int_0^\infty dq \int_{|k-q|}^{|k+q|} dp$. By using the relation (25) in equation (24), we finally arrive at

$$\left[\frac{d}{dt} + 2\nu k^2 \right] \mathcal{U}^X = S_{d-2} \int_{\Delta_k} dq dp \left(\frac{pq}{k} \right)^{d-2} (1-x^2)^{\frac{(d-3)}{2}} \mathcal{T}^X(\Delta_k). \quad (26)$$

Here S_d is the solid angle of a d -dimensional sphere. Now we have to integrate equation (26) over the $p-q$ plane that can form a triangle with a side of length $k = |\mathbf{k}|$. By using the triangle inequality, this region would involve $p+q < k < |p-q|$, as shown in figure 4, for every k . Isotropy implies that the energy spectrum and spectral energy are related by

$$\mathcal{U}^X(k) = \frac{2E^X(k)}{k^{d-1} S_{d-1}} \quad (27)$$

All the geometric coefficients are not independent. In fact it is not difficult to prove the following constraints:

$$2a_{kpq} = b_{kpq} + b_{kqp} \quad (28a)$$

$$2a_{kpq} = c_{kpq} + c_{kqp} \quad (28b)$$

$$k^2 b_{kpq} = p^2 b_{pkq} \quad (28c)$$

$$k^2 h_{kpq} = p^2 h_{pkq} \quad (28d)$$

$$g_{kpq} = h_{kpq} + \frac{q^2}{k^2} c_{qkp} \quad (28e)$$

$$q^4 c_{qkp}^{(3)} = k^4 c_{kqp}^{(3)} \quad (28f)$$

$$(p^2 c_{kqp} + q^2 c_{kpq}) = (p^2 h_{kqp} + q^2 h_{kpq}) \frac{(d-2)}{(d-1)} \quad (28g)$$

$$k^2 a_{kpq}^{(2)} = p^2 b_{kpq}^{(2)} + q^2 b_{kqp}^{(2)} \quad (28h)$$

$$k^2 c_{kqp}^{(2)} = q^2 \left(h_{pqk}^{(2)} - h_{qpk}^{(2)} \right). \quad (28i)$$

With the superscripts (2) and (3) corresponding to two and three dimensions, respectively. Finally, exploiting the symmetry between p and q , we construct the quasi-normal MHD equations:

$$\left[\frac{d}{dt} + 2\nu k^2 \right] E^u(k, t) = 8K_d \int_{\Delta_k} dq \, dp \left[T_u^{uu}(k, p, q) + T_u^{bb}(k, p, q) + T_u^{ub}(k, p, q) \right] W_d(\Delta_k) \quad (29a)$$

$$\left[\frac{d}{dt} + 2\eta k^2 \right] E^b(k, t) = 8K_d \int_{\Delta_k} dq \, dp \left[T_b^{ub}(k, p, q) + T_b^{bb}(k, p, q) \right] W_d(\Delta_k) \quad (29b)$$

$$T_u^{uu}(k, p, q) = \hat{\theta}(\omega_{kpq}^u) \frac{k}{pq} b_{kpq} [k^{d-1} E^u(p) - p^{d-1} E^u(k)] E^u(q) \quad (29c)$$

$$T_u^{bb}(k, p, q) = \hat{\theta}(\omega_{kpq}^b) \frac{k}{pq} c_{kpq} k^{d-1} E^b(p) E^b(q) \quad (29d)$$

$$T_u^{ub}(k, p, q) = -\hat{\theta}(\omega_{kpq}^b) \frac{k}{pq} c_{kpq} p^{d-1} E^u(k) E^b(q) \quad (29e)$$

$$T_b^{ub}(k, p, q) = \hat{\theta}(\omega_{kpq}^b) \frac{k}{pq} h_{kpq} [k^{d-1} E^b(p) - p^{d-1} E^b(k)] E^u(q) + \hat{\theta}(\omega_{pqk}^b) \frac{p}{kq} c_{pqk} k^{d-1} E^u(p) E^b(q) \quad (29f)$$

$$T_b^{bb}(k, p, q) = -\hat{\theta}(\omega_{pqk}^b) \frac{p}{kq} c_{pqk} p^{d-1} E^b(k) E^b(q). \quad (29g)$$

The dimensional pre-factor K_d and the triad weights $W_d(\Delta_k)$ are

$$K_d = \frac{1}{(d-1)^2} \frac{S_{d-2}}{S_{d-1}} \quad (30a)$$

$$W_d(\Delta_k) = \left(\frac{\sin^2 x}{k^2} \right)^{\frac{(d-3)}{2}}. \quad (30b)$$

This is a closed set of integro-differential equations, that can be solved numerically. Note that the operators $\hat{\theta}(\omega_{kpq}^u)$ and $\hat{\theta}(\omega_{kpq}^b)$ involve time integrals, as defined in equation (12). This QN model (equation (29)) respects the conservation laws that the original PDE has. The sum of kinetic and magnetic energy is conserved for the ideal fluid, that is $\nu = \eta = 0$. Particularly, in two-dimensions, the net magnetic potential is conserved and for a pure kinetic model ($\mathbf{B} = 0$) the enstrophy remains conserved.

Appendix B. The eddy-damped quasi-normal approximation

We now make two further approximations in the spirit of the EDQNM model [29, 33, 48, 49] for hydrodynamic turbulence. This is because quasi Normal by itself does not ensure the positive definite nature of kinetic energy owing to divergences in the third-order moments. This is cured by *damping*, which ensures a saturation of the third-order moments by introducing an inverse time-scale μ_{kpq} for the triad k, p, q ,

$$\left[\frac{d}{dt} + \nu(k^2 + p^2 + q^2) + \mu_{kpq} \right] \langle uuu \rangle = \langle uu \rangle \langle uu \rangle, \quad \mu_{kpq} = \mu_k + \mu_p + \mu_q, \quad (31)$$

which ought to originate from the spectrum. On dimensional grounds this is simply

$$\mu_k \approx [k^3 E^u(k)]^{\frac{1}{2}}. \quad (32)$$

Further improvement of this

$$\mu_k = \alpha_d \left[\int_0^k dp p^2 E^u(p) \right]^{\frac{1}{2}} \quad (33)$$

factors in the deformation of eddies of size k^{-1} by larger eddies. This allows for a free parameter α_d which, for fluid turbulence, fixes the d -dimensional Kolmogorov constant C_d for the corresponding d -dimensional kinetic energy spectrum $E(k) = C_d k^{-5/3} \epsilon^{2/3}$. For a given d , C_d is chosen from interpolating the values given in [31].

We use the same inspiration, for our closure model, to define (for the magnetic field)

$$\mu_k = \alpha_d \left[\int_0^k dp p^2 E^b(p) \right]^{\frac{1}{2}} + \sqrt{\frac{2}{3}} \left(\int_0^k dp E^b(p) \right)^{\frac{1}{2}} \quad (34)$$

with the additional piece accounting for the effects of the Alfvén waves [8]; the coefficient $\sqrt{2/3}$ comes from an explicit calculation of the Alfvén timescales for a Gaussian large-scale magnetic fields [34].

In summary, the final eddy-damping time-scales, acting linearly on third-order moments in our closure model, are given by:

$$\begin{aligned} \omega_k^u &= \nu k^2 + \alpha_d \left[\int_0^k dp p^2 E^V(p) \right]^{\frac{1}{2}} ; \\ \omega_k^b &= \eta k^2 + \alpha_d \left[\int_0^k dp p^2 E^b(p) \right]^{\frac{1}{2}} + \sqrt{\frac{2}{3}} \left(\int_0^k dp E^b(p) \right)^{\frac{1}{2}} . \end{aligned} \quad (35)$$

Appendix C. The eddy-damped quasi-normal Markovian model

The eddy-damping time-scales by itself does not guarantee the positive definiteness of the the energy spectrum. A final approximation, due to Orszag [29], is *Markovianization*. This assumes that the third-order moments vary slowly when compared to the exponential decay in the $\hat{\theta}(\omega_{kpq}^u)$ operator. This separation of time-scales allows the approximation, where the time integral in $\hat{\theta}(\omega_{kpq}^u)$ is computed explicitly, leading to further simplification

$$\hat{\theta}(\omega_{kpq}^u) = \int_0^t ds \exp(-\omega_{kpq}^u(t-s)) = \frac{1 - \exp(-\omega_{kpq}^u t)}{\omega_{kpq}^u}. \quad (36)$$

Further, in the large t limit, $\hat{\theta}(\omega_{kpq}^u) = 1/\omega_{kpq}^u \equiv \theta_{kpq}^u$ and $\hat{\theta}(\omega_{kpq}^b) = 1/\omega_{kpq}^b \equiv \theta_{kpq}^b$. This establishes an instantaneous relationship between third and second order moments and makes the process memoryless or Markovian.

We are now in a position, with all our ingredients in place, to write the final set of d -dimensional, MHD-EDQNM equations for incompressible, MHD turbulence:

$$\frac{d}{dt} E^u(k, t) = T^u(k, t) - 2\nu k^2 E^u(k, t); \quad (37a)$$

$$\frac{d}{dt} E^b(k, t) = T^b(k, t) - 2\eta k^2 E^b(k, t). \quad (37b)$$

It is extremely useful to study how the kinetic energy and magnetic energy spectrum interacts in the transfer terms. In order to do that, divide the contributions to the transfer terms, as done in the main text, as *self* (subscript (s)) and *coupled* (subscript (c)):

$$T^u(k) = T_{(s)}^u(k) + T_{(c)}^u(k); \quad (38a)$$

$$T^b(k) = T_{(s)}^b(k) + T_{(c)}^b(k); \quad (38b)$$

where

$$T_{(s)}^u(k) = 8K_d \int_{\Delta_k} dq dp W_d(\Delta_k) \theta_{kpq}^u \frac{k}{pq} b_{kpq} [k^{d-1} E^u(p) - p^{d-1} E^u(k)] E^u(q) \quad (39a)$$

$$T_{(c)}^u(k) = 8K_d \int_{\Delta_k} dq dp W_d(\Delta_k) \theta_{kpq}^b \frac{k}{pq} c_{kpq} [k^{d-1} E^b(p) - p^{d-1} E^b(k)] E^b(q) \quad (39b)$$

$$T_{(s)}^b(k) = 8K_d \int_{\Delta_k} dq dp W_d(\Delta_k) \theta_{kpq}^b \frac{k}{pq} h_{kpq} [k^{d-1} E^b(p) - p^{d-1} E^b(k)] E^u(q) \quad (39c)$$

$$T_{(c)}^b(k) = 8K_d \int_{\Delta_k} dq dp W_d(\Delta_k) \theta_{kpq}^u \frac{p}{kq} c_{kpq} [k^{d-1} E^u(p) - p^{d-1} E^u(k)] E^b(q). \quad (39d)$$

Appendix D. Details of the numerical simulations of the d -dimensional MHD-EDQNM model and benchmarks

Given that the original EDQNM-MHD equations have infinite degrees of freedom, to numerically study them, we have to discretize the wavenumber space, say $\mathcal{D}$. Since the EDQNM-MHD equations produce smooth function for the energy spectrum, logarithmic discretization of wavenumber space allows us to achieve high Reynolds number (magnetic and kinetic) in the numerical simulation. We discretize the wavenumbers as follows:

$$\mathcal{D} \equiv \{k_i = k_1 \lambda^{i-1}\}, i = 1, 2, \dots, N. \quad (40)$$

The wavenumber bands are chosen as $\Delta k_i = k_i \ln \lambda$. Suppose we denote the upper k_i^+ and lower k_i^- limits of the i th band by

$$k_i^+ = k_i + \Delta k_i^+ \quad (41a)$$

$$k_i^- = k_i - \Delta k_i^- \quad (41b)$$

$$\Delta k_i^+ + \Delta k_i^- = \Delta k_i. \quad (41c)$$

Since we wish to cover the whole of the wavenumber space till k_N without any gaps or overlaps, the lower limit of the $(i+1)$ th band should coincide with the upper limit of the i th band:

$$k_i + \Delta k_i^+ = k_{i+1} - \Delta k_{i+1}^- \quad (42a)$$

$$\Delta k_{i+1}^- = \Delta k_i^- + k_{i+1} \left(1 - \frac{1}{\lambda} - \frac{\ln \lambda}{\lambda}\right) \quad (42b)$$

$$= \Delta k_i^- + k_1 \left(1 - \frac{1}{\lambda} - \frac{\ln \lambda}{\lambda}\right) \frac{\lambda^i - \lambda}{\lambda - 1} \quad (42c)$$

$$k_i^+ = \frac{\Delta k_i}{\lambda - 1} + \frac{k_1}{\lambda - 1} (\lambda - 1 - \ln \lambda) - \Delta k_1^- \quad (42d)$$

$$k_i^- = \frac{\Delta k_{i-1}}{\lambda - 1} + \frac{k_1}{\lambda - 1} (\lambda - 1 - \ln \lambda) - \Delta k_1^-. \quad (42e)$$

Now, without loss of generality, we choose $\Delta k_1^- = k_1 \frac{(\lambda - 1 - \ln \lambda)}{\lambda - 1}$, whence

$$k_i^- = \frac{\Delta k_i}{\lambda - 1} \quad (43a)$$

$$k_i^+ = \frac{\Delta k_{i+1}}{\lambda - 1}. \quad (43b)$$

In this framework of discrete wavenumber bands, the energy integral is straightforward, that is

$$\int dk f(k) \equiv \sum_{i=1}^N \Delta k_i f(k_i) \quad (44)$$

and the integral in the transfer term over the triad Δ_k is approximated as:

$$\int_{\Delta k_i} dq dp F(k_i, q, p) \equiv \sum_{j=1}^N \sum_{l=l_{\min}}^{l_{\max}} \Delta k_j \Delta k_l F(k_i, k_j, k_l) \quad (45a)$$

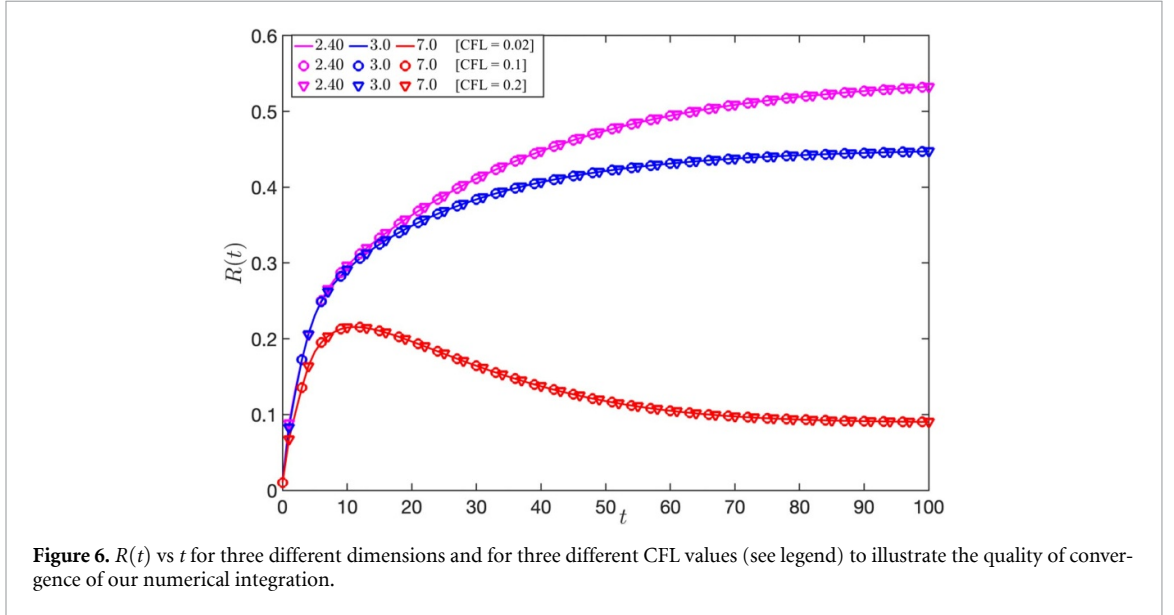
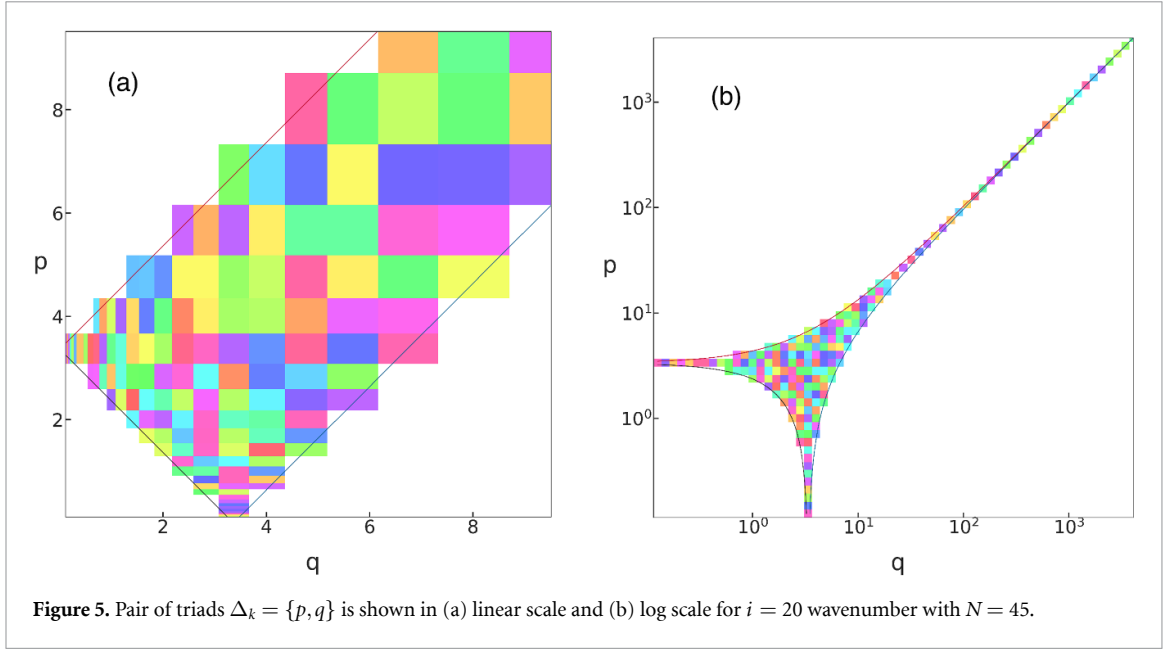
$$l_{\min}(i, j) = [\text{Log}_\lambda(|k_i - k_j|)]_{>} + 1 \quad (45b)$$

$$l_{\max}(i, j) = \text{Max} \left\{ N, [\text{Log}_\lambda(k_i + k_j)]_{<} + 1 \right\}. \quad (45c)$$

The integration limits are chosen such that $k_i, k_j, k_l \in \mathcal{D}$ can form a triangle. In the above equations, $[x]_{<}$ and $[x]_{>}$ correspond to the lowest and the greatest integer function.

The visualization of the integral in equation (45) is shown in figure 5 for $i = 20$ with $N = 45$.

With the Fourier lattice established, we can now solve the d -dimensional EDQNM-MHD equations with the triads calculated as discussed above. We perform simulations with $N = 25, 45$ and 53 which results in a total number of triads 3067, 7317, and 9353, respectively. We have checked that our results are insensitive, beyond the obvious extension of the spectral range, to the number of modes being simulated for all our dimensions. In the manuscript we naturally report results for $N = 53$ and 9353 triads.



We use a forcing spectrum $F(k)$, concentrated at large scales via $F(k) \sim k^2 \exp(-k^2/2k_I^2)$, with $k_I = 1$ setting the injection scale, on the fluid velocity field to drive and maintain the fluid in a non-equilibrium stationary state. The forcing scale is also a convenient way to define our magnetic Reynolds number $Rm = \frac{\sqrt{\mathcal{E}^u}}{k_I \eta}$ for our simulations. We tune our forcing to ensure the fluid kinetic energy is maintained at $\mathcal{E}^u = 1$ and, since $k_I = 1$, the magnetic Reynolds number reduces to simply $Rm = 1/\eta$.

For time marching, we use explicit methods for treating the non-linear transfer terms and an implicit time integration for the linear viscous and diffusion terms to ensure stability. We have used a fourth-order Runge–Kutta scheme as well as a fourth-order Adams–Bashforth scheme to integrate in time and found that our results to be independent of the specific choice of our integration scheme; for the results presented in the manuscript, we use the Adams–Bashforth scheme.

Finally, we ensure that the Courant–Freidrich–Levy (CFL) condition is always satisfied by explicitly checking for the CFL at every iteration. Since EDQNM equations are in spectral space, we define the CFL condition, heuristically as

$$\text{CFL} := \frac{\delta t}{t_{\min}} < 1 \quad (46)$$

where $t_{\min} = \text{Min} \left\{ \frac{1}{\sqrt{\varepsilon^u} k_{\max}}, \frac{1}{\sqrt{\varepsilon^v} k_{\max}}, \frac{1}{\nu k_{\max}^2}, \frac{1}{\eta k_{\max}^2} \right\}$ assures that the time-step is smaller than the smallest time-scale in the dynamics. In particular, we have checked that our results converge for 3 different values of the CFL, namely 0.2, 0.1, and 0.02, corresponding to a $\delta t = 2 \times 10^{-4}$, 10^{-4} and 2×10^{-5} , respectively for $Pr = 1$ and $Rm = 2000$; for other Prandtl and magnetic Reynolds numbers the time step is adjusted to have the same values of CFL.

We show the quality of numerical convergence by providing in figure 6(a) linear plot of $R(t)$ vs t for 3 different dimensions and for 3 different values of CFL (see legend) for $Pr = 1$. As is clear from the figure, the data from all three simulations and for the three different dimensions converge exactly. In the manuscript, we report results for $CFL = 0.02$.

Appendix E. No-dynamo-dynamo transition for $Pr \neq 1$

We perform additional simulations with $0.01 \leq Pr \leq 5$ to confirm the robustness of our result of the existence of the lower and upper critical dimensions. In figure 7 we show the resultant phase diagram plots for $Pr = 0.01$ and $Pr = 5$. The lower and upper critical dimensions indeed change as a function of Pr but, importantly, these results confirm their existence. We leave to future studies about how exactly these dimensions depend on the Reynolds and Prandtl numbers.

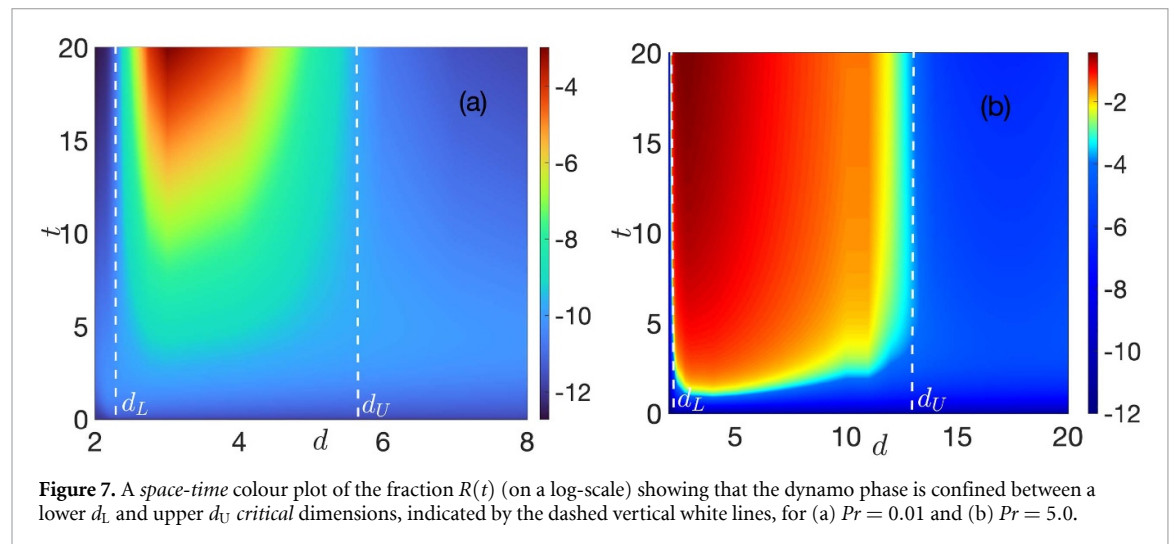


Figure 7. A space-time colour plot of the fraction $R(t)$ (on a log-scale) showing that the dynamo phase is confined between a lower d_L and upper d_U critical dimensions, indicated by the dashed vertical white lines, for (a) $Pr = 0.01$ and (b) $Pr = 5.0$.

ORCID iDs

Sugan Durai Murugan  0000-0003-2698-6851

Giorgio Krstulovic  0000-0002-9934-6292

Dario Vincenzi  0000-0003-3332-3802

Samriddhi Sankar Ray  0000-0001-9407-0007

References

- [1] Ogilvie G I 2016 *J. Plasma Phys.* **82** 205820301 (available at: www.jetp.ras.ru/cgi-bin/e/index/e/6/6/p1184?a=list)
- [2] Brandenburg A 2018 *J. Plasma Phys.* **84** 735840404
- [3] Brandenburg A and Ntormousi E 2023 *Annu. Rev. Astron. Astrophys.* **61** 561–606
- [4] Moffatt K and Dormy E 2019 *Self-Exciting Fluid Dynamos* (Cambridge University Press)
- [5] Rincon F 2019 *J. Plasma Phys.* **85** 205850401
- [6] Tobias S M 2021 *J. Fluid Mech.* **912** 1
- [7] Choudhuri A R 1998 *The Physics of Fluids and Plasmas: an Introduction for Astrophysicists* (Cambridge University Press) (available at: <https://www.cambridge.org/core/books/physics-of-fluids-and-plasmas/8A235D6F1D9DA51F05237D42BDFEFD06>)
- [8] Davidson P A 2001 *An Introduction to Magnetohydrodynamics* (Cambridge University Press) (available at: www.cambridge.org/core/books/an-introduction-to-magnetohydrodynamics/7FBA835C19E375799D921ED81202F91F)
- [9] Biskamp D 2003 *Magnetohydrodynamic Turbulence* (Cambridge University Press) (available at: www.cambridge.org/core/books/magnetohydrodynamic-turbulence/454322A082527536DCEEEDFB2FA8B985)
- [10] Galtier S 2016 *Introduction to Modern Magnetohydrodynamics* (Cambridge University Press)

- [11] Schekochihin A A 2022 *J. Plasma Phys.* **88** 155880501
- [12] Monchaux R *et al* 2007 *Phys. Rev. Lett.* **98** 044502
- [13] Kazantsev A P 1968 *Sov. J. Exp. Theor. Phys.* **26** 1031 (available at: <https://ui.adsabs.harvard.edu/abs/1968JETP...26.1031K>)
- [14] Falkovich G, Gawedzki K and Vergassola M 2001 *Rev. Mod. Phys.* **73** 913–75
- [15] Brandenburg A and Subramanian K 2005 *Phys. Rep.* **417** 1–209
- [16] Gruzinov A, Cowley S and Sudan R 1996 *Phys. Rev. Lett.* **77** 4342–5
- [17] Schekochihin A A, Boldyrev S A and Kulsrud R M 2002 *Astrophys. J.* **567** 828
- [18] Arponen H and Horvai P 2007 *J. Stat. Phys.* **129** 205–39
- [19] Martins Afonso M, Mitra D and Vincenzi D 2019 Proceedings of the royal society a: mathematical *Physical and Engineering Sciences* **475** 20180591
- [20] Wilson K G and Fisher M E 1972 *Phys. Rev. Lett.* **28** 240–3
- [21] Fournier J D and Frisch U 1978 *Phys. Rev. A* **17** 747–62
- [22] L'vov V S, Pomyalov A and Procaccia I 2002 *Phys. Rev. Lett.* **89** 064501
- [23] Celani A, Musacchio S and Vincenzi D 2010 *Phys. Rev. Lett.* **104** 184506
- [24] Frisch U, Pomyalov A, Procaccia I and Ray S S 2012 *Phys. Rev. Lett.* **108** 074501
- [25] Ray S S 2015 *Pramana* **84** 395–407
- [26] Ray S S 2018 *Phys. Rev. Fluids* **3** 072601
- [27] Picardo J R, Bhatnagar A and Ray S S 2020 *Phys. Rev. Fluids* **5** 042601
- [28] Kraichnan R H 1967 *Phys. Fluids* **10** 1417–23
- [29] Orszag S A 1970 *J. Fluid Mech.* **41** 363–86
- [30] Rose H A and Sulem P L 1978 *J. Physique* **39** 441–84
- [31] Clark D, Ho R D J G and Berera A 2021 *J. Fluid Mech.* **912** A40
- [32] Clark D, Armua A, Ho R D J G and Berera A 2022 *J. Fluid Mech.* **930** A17
- [33] Lesieur M 2008 *Turbulence in Fluids* (Springer) (<https://hal.archives-ouvertes.fr/hal-00261553>)
- [34] Pouquet A, Frisch U and Leorat J 1976 *J. Fluid Mech.* **77** 321–54
- [35] Schilling O and Zhou Y 2002 *J. Plasma Phys.* **68** 389–406
- [36] Pouquet A 1978 *J. Fluid Mech.* **88** 1–16
- [37] Frisch U, Lesieur M and Sulem P L 1976 *Phys. Rev. Lett.* **37** 895–7
- [38] Zel'dovich I B 1957 *Sov. Phys. JETP* **4** 460–2 (available at: <https://www.jetp.ras.ru/cgi-bin/e/index/e/6/6/p1184?a=list>)
- [39] Zel'dovich Y B and Ruzmaikin A A 1980 *Sov. Phys. JETP* **51** 493–7
- [40] Kolokolov I V 2017 *J. Phys. A: Math. Theor.* **50** 155501
- [41] Burden A D 1991 Towards an EDQNM-Closure for Inhomogeneous Turbulence *Advances in Turbulence 3* ed A V Johansson and P H Alfredsson (Springer) pp 387–94
- [42] Frederiksen J S 2012 *Entropy* **14** 32–57
- [43] Frederiksen J S and O'Kane T J 2023 *Atmosphere* **14** 1098
- [44] Briard A and Gomez T 2017 *J. Fluid Mech.* **821** 539–81
- [45] Briard A and Gomez T 2018 *J. Plasma Phys.* **84** 905840110
- [46] Turner L and Pratt J 2002 *J. Phys. A: Math. Gen.* **35** 781–93
- [47] Vincenzi D 2002 *J. Stat. Phys.* **106** 1073–91
- [48] Sagaut P 2002 *Edqnm Modeling Large Eddy Simulation for Incompressible Flows: An Introduction* ed P Sagaut (Springer) pp 391–5
- [49] Sagaut P and Cambon C 2018 *Homogeneous Turbulence Dynamics* (Springer)