Energy spectrum of two-dimensional acoustic turbulence: Supplemental Material

Adam Griffin,[1] Giorgio Krstulovic,[2] Victor L'vov,[3] and Sergey Nazarenko[1]

[1] *Institut de Physique de Nice, Université Côte d'Azur, CNRS, Nice, France*

[2] *Université Côte d'Azur, Observatoire de la Côte d'Azur, CNRS, Laboratoire Lagrange, Boulevard de l'Observatoire CS 34229 - F 06304 NICE Cedex 4, France.*

[3] *Dept. of Chemical and Biological Physics, Weizmann Institute of Science, Rehovot 76100, Israel*

(Dated: April 8, 2022)

I. HAMILTONIAN FORMULATION OF ACOUSTIC TURBULENCE

In this section we review the Hamiltonian formulation of acoustic turbulence and obtain the interaction term represented by V^k_{12} in Eq.(5) of the main text. In the acoustic limit, this term is well known to be $V^k_{12} = V_0\sqrt{kk_1k_2}$ [1–4], however the constant V_0 in these references takes different values. Here, we will carefully derive its value.

The starting point is the action (per unit of mass) for a compressible, isentropic, irrotational fluid [5] that reads:

$$\mathcal{S} = \frac{1}{\rho_0 L^2}\int dt d^2x \left[-\rho\dot{\phi} - \frac{\rho}{2}(\nabla\phi)^2 - \frac{c_\mathrm{s}^2}{2\rho_0}(\rho-\rho_0)^2\right], \tag{1}$$

where ρ_0 is the bulk density and c_s, as it will be clear later, is the speed of sound. Note that the dimensions of the fields are $[\phi] = L^2/T$ and $[\rho] = M/L^2$.

Varying the action with respect to ρ and ϕ and we obtain the fluid equations,

$$\dot{\rho} + \nabla\left(\rho\nabla\phi\right) = 0, \tag{2}$$

$$\dot{\phi} + \frac{1}{2}\nabla\phi^2 = -\frac{c_\mathrm{s}^2}{\rho_0}(\rho-\rho_0). \tag{3}$$

Acoustic waves are readily obtained by linearizing the equations about $\phi = 0$ and $\rho = \rho_0$, which leads to the wave equation $\ddot{\phi} = c_\mathrm{s}^2\nabla^2\phi$.

Note that the action (1) is not written in a Hamiltonian way. Making the following change of variables $\rho = \rho_0 A^2$ and $p = 2A\phi$, after substituting in (1), we obtain

$$\mathcal{S} = \int dt\frac{d^2x}{L^2}\frac{1}{2}\left(A\dot{p} - \dot{A}p\right) - \int dtH \quad \text{with} \quad H = \int\frac{d^2x}{L^2}\left[\frac{1}{8}\left(\nabla p - \frac{p\nabla A}{A}\right)^2 + \frac{c_\mathrm{s}^2}{2}(A^2-1)^2\right], \tag{4}$$

where the equation of motion are now given by

$$\dot{p} = \frac{\delta H}{\delta A}, \quad \dot{A} = -\frac{\delta H}{\delta p}. \tag{5}$$

We remark that the units of the new fields are $[A] = 1$ and $[p] = L^2/T$. The Hamiltonian per unit of mass has units $[H] = L^2/T^2$ as usual in hydrodynamics.

A. Acoustic waves

Waves are obtained by making $A \to 1 + \tilde{A}$ and $p \to \tilde{p}$. Dropping tildes and keeping the terms up to the cubic order, we rewrite the Hamiltonian as $H = H_2 + H_3$, where the second and third order terms are

$$H_2 = \int\frac{d^2x}{L^2}\left[\frac{1}{8}(\nabla p)^2 + 2c_\mathrm{s}^2A^2\right], \quad H_3 = \int\frac{d^2x}{L^2}\left[2c_\mathrm{s}^2A^3 - \frac{1}{4}p(\nabla p)\cdot(\nabla A)\right]. \tag{6}$$

We now assume that the fields are periodic and write them as $p(\mathbf{x}) = \sum_\mathbf{k} p_\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}}$ and $A(\mathbf{x}) = \sum_\mathbf{k} A_\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}}$. The Hamiltonian and the action become:

$$\mathcal{S} = \int dt\sum_\mathbf{k}\frac{1}{2}\left(A_k\dot{p}^*_\mathbf{k} - \dot{A}_\mathbf{k}p^*_\mathbf{k}\right) - \int dt(H_2 + H_3), \tag{7}$$

$$H_2 = \sum_\mathbf{k}\frac{1}{8}k^2|p_\mathbf{k}|^2 + 2c_\mathrm{s}^2|A_\mathbf{k}|^2, \tag{8}$$

$$H_3 = \sum_{1,2,3} 2c_\mathrm{s}^2A_1A_2A_3\delta_{1,2,3} + \frac{1}{4}p_1p_2A_3\mathbf{k}_2\cdot\mathbf{k}_3\delta_{1,2,3}, \tag{9}$$

where $\delta_{1,2,3}$ is 1 if $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0$, and 0 otherwise.

In order to write the Hamiltonian and the action in the canonical form, we perform the following change of variables

$$p_{\mathbf{k}} = i\frac{1}{\sqrt{2}}\left(\frac{\alpha}{k^2}\right)^{\frac{1}{4}}\left(a_{\mathbf{k}} - a^*_{-\mathbf{k}}\right), \tag{10}$$

$$A_{\mathbf{k}} = \frac{1}{\sqrt{2}}\left(\frac{k^2}{\alpha}\right)^{\frac{1}{4}}\left(a_{\mathbf{k}} + a^*_{-\mathbf{k}}\right), \tag{11}$$

where $\alpha = 16c_{\mathrm{s}}^2$. The value of this coefficient is set in order to kill the off-diagonal terms in H_2. At the leading order, the action becomes

$$\mathcal{S}_2 = \int dt \frac{i}{2}\sum_{\mathbf{k}}\left(\dot{a}_{\mathbf{k}}a^*_{\mathbf{k}} - a_{\mathbf{k}}\dot{a}^*_{\mathbf{k}}\right) - \int dt H_2\,, \quad \text{with } H_2 = \sum_{\mathbf{k}} c_{\mathrm{s}} k |a_{\mathbf{k}}|^2 = \sum_{\mathbf{k}} \omega_k |a_{\mathbf{k}}|^2. \tag{12}$$

Then,

$$\frac{\delta \mathcal{S}_2}{\delta a^*} = 0 \implies i\dot{a}_{\mathbf{k}} = \frac{\partial H}{\partial a^*_{\mathbf{k}}} = \omega_k a_{\mathbf{k}}. \tag{13}$$

B. H_3 terms

The cubic part of the Hamiltonian requires some tedious work. Keeping only resonant terms we obtain the following contributions

$$\sum A_1 A_2 A_3 \delta_{1,2,3} = \frac{1}{2^{3/2}}\sum_{1,2,3}\frac{\sqrt{k_1k_2k_3}}{\alpha^{\frac{3}{4}}}(a_1+a^*_{-1})(a_2+a^*_{-2})(a_3+a^*_{-3}) = \frac{3}{2^{3/2}}\sum\frac{\sqrt{k_1k_2k_3}}{\alpha^{\frac{3}{4}}}\left(a_1a_2^*a_3^* + a_1^*a_2a_3\right)\delta^1_{2,3}, \tag{14}$$

where $\delta^1_{2,3} = \delta_{-1,2,3}$. The second term requires more manipulations

$$\sum p_1p_2A_3\mathbf{k}_2\cdot\mathbf{k}_3\delta_{1,2,3} = -\frac{1}{2^{3/2}}\sum_{1,2,3}\frac{\alpha^{\frac{1}{4}}}{\sqrt{k_1k_2k_3}}k_3(\mathbf{k}_2\cdot\mathbf{k}_3)(a_1-a^*_{-1})(a_2-a^*_{-2})(a_3+a^*_{-3})\delta_{1,2,3} \tag{15}$$

$$= -\frac{1}{2\times 2^{3/2}}\sum_{1,2,3}\frac{\alpha^{\frac{1}{4}}}{\sqrt{k_1k_2k_3}}k_3(\mathbf{k}_1\cdot\mathbf{k}_3+\mathbf{k}_2\cdot\mathbf{k}_3)(a_1-a^*_{-1})(a_2-a^*_{-2})(a_3+a^*_{-3})\delta_{1,2,3} \tag{16}$$

$$= \frac{1}{2\times 2^{3/2}}\sum_{1,2,3}\frac{\alpha^{\frac{1}{4}}}{\sqrt{k_1k_2k_3}}k_3^3(a_1-a^*_{-1})(a_2-a^*_{-2})(a_3+a^*_{-3})\delta_{1,2,3}, \tag{17}$$

where form the second to third line we used the resonant condition $\mathbf{k}_1+\mathbf{k}_2=-\mathbf{k}_3$. Again, keeping only resonant terms, changing summation variables and using symmetries, we can replace inside the sum

$$k_3^3(a_1-a^*_{-1})(a_2-a^*_{-2})(a_3+a^*_{-3})\delta_{1,2,3} \to \left(a_1a_2^*a_3^*+a_1^*a_2a_3\right)(k_1^3-2k_3^3)\delta^1_{2,3} \to \left(a_1a_2^*a_3^*+a_1^*a_2a_3\right)(k_1^3-k_2^3-k_3^3)\delta^1_{2,3} \tag{18}$$

Finally, using the resonant condition $k_1=k_2+k_3$, we have $k_1^3-k_2^3-k_3^3=(k_2+k_3)^3-k_2^3-k_3^3=3(k_2+k_3)k_2k_3=3k_1k_2k_3$. Gathering all the terms

$$H_3 = \frac{1}{2^{3/2}}\sum(a_1a_2^*a_3^*+c.c)\delta^1_{2,3}\sqrt{k_1k_2k_3}\left[2c_{\mathrm{s}}^2\frac{3}{\alpha^{3/4}}+\frac{1}{4}\frac{3\alpha^{\frac{1}{4}}}{2}\right] = \sum V^1_{23}(a_1a_2^*a_3^*+c.c)\delta^1_{2,3}, \tag{19}$$

where $V^1_{23} = V_0\sqrt{k_1k_2k_3}$, with

$$V_0 = \frac{1}{2^{3/2}}\left[2c_{\mathrm{s}}^2\frac{3}{\alpha^{3/4}}+\frac{1}{4}\frac{3\alpha^{\frac{1}{4}}}{2}\right] = \frac{3}{4\sqrt{2}}\sqrt{c_{\mathrm{s}}}, \tag{20}$$

the formula used to obtained Eq. (18) from Eq. (16) in the main text.

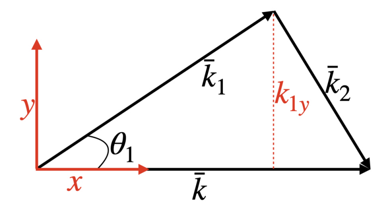

FIG. 1.
Wave vector triad. We choose a coordinate system such that $\bar{k}$ is aligned with the x-axis. θ_1 is the angle between $\bar{k}$ and $\bar{k}_1$, k_{1y} is the y-component of $\bar{k}_1$.

II. ANALYSIS OF THE COLLISION TERM

A. Analysis of $\mathcal{R}^k_{12}$ term

According to Eqs. (3) of the main text, the first term in the collision integral ($\mathcal{R}^k_{12}$) contains

$$\Delta^{\boldsymbol{k}}_{\boldsymbol{12}} \equiv \int_{\boldsymbol{k}_2, k_{1y}} \delta(\omega^k_{12}) \boldsymbol{\delta}^{\boldsymbol{k}}_{\boldsymbol{12}} d\boldsymbol{k}_1 d\boldsymbol{k}_2 \approx \frac{k_1 k_2 \, dk_{1x}}{c_{\rm s}\, k\, |k_{1y}|} \ . \tag{21}$$

From Fig. 1 we can see that to leading order

$$k_{1y} = k_1 \sin(\theta_{1k}) \approx k_1 \theta_{1k}. \tag{22}$$

To find θ_{1k} consider the wave number resonance condition,

$$\begin{aligned} \boldsymbol{k}_2 = \boldsymbol{k} - \boldsymbol{k}_1 \Rightarrow k_2^2 = k_1^2 + k^2 - 2\boldsymbol{k}_1 \cdot \boldsymbol{k} = k_1^2 + k^2 - 2k_1 k \cos\theta_{1k} \approx (k - k_1)^2 + k k_1 \theta_{1k}^2 \\ \Rightarrow k_2 \approx (k - k_1) + \frac{k k_1}{2k_2}\, \theta_{1k}^2 \ . \end{aligned} \tag{23}$$

Next, consider the frequency resonance condition,

$$k_1 + k_2 - k = -a^2 (k_1^3 + k_2^3 - k^3), \tag{24}$$

and substitute k_2 from Eq. (23) to the LHS and $k_2 = (k - k_1)$ to the RHS of this equation. This gives $\dfrac{k k_1 \theta_{k1}^2}{2k_2} = 3a^2 k k_1 k_2$ or $\theta_{k1}^2 = 6a^2 k_2^2$ and

$$k_{1y} = \sqrt{6} a\, k_1 k_2 \ . \tag{25}$$

Together with Eq. (21), this finally gives

$$\Delta^{\boldsymbol{k}}_{\boldsymbol{12}} \approx \frac{dk_{1x}}{\sqrt{6}\, c_{\rm s}\, ak} = \frac{\delta(k - k_1 - k_2) dk_1 dk_2}{\sqrt{6}\, c_{\rm s}\, ak}, \tag{26}$$

also shown in Eq. (8.b) in the main text. Here, we have replaced dk_{1x} by dk_1 and inserted $\delta(k - k_1 - k_2) dk_2 = 1$ to stress that $k = k_1 + k_2$ in the used approximation.

B. Contribution to $\mathcal{R}^1_{k2}$ and $\mathcal{R}^2_{k1}$

Similar derivations using the wave number and frequency resonance conditions lead to

$$\Delta^{\boldsymbol{1}}_{\boldsymbol{2k}} \approx \frac{dk_{1x}}{\sqrt{6}\, c_{\rm s}\, ak} = \frac{\delta(k_1 - k - k_2) dk_1 dk_2}{\sqrt{6}\, c_{\rm s}\, ak} \quad \text{and} \quad \Delta^{\boldsymbol{2}}_{\boldsymbol{1k}} \approx \frac{\delta(k_2 - k - k_1) dk_1 dk_2}{\sqrt{6}\, c_{\rm s}\, ak} \ . \tag{27}$$

Together, Eqs. (26) and (27) lead to the collision integral (10) of the main text.

C. Proof of the interaction locality

Convergence of the integral in Eq. (10) of the main text is referred to as the interaction locality property. First, we note that this integral is trivially convergent for $x = 1$ because the integrand is identically equal to zero. This exponent corresponds to the thermodynamic energy equipartition state, i.e. a trivial zero-flux equilibrium which we will not be interested in. Thus, below we will consider the cases with $x \neq 1$.

1. Infrared locality

Consider first the infra-red (IR) locality, i.e. convergence of the integral in Eq. (10) of the main text, in the region $k_1 \ll k$. We take into account that for the acoustic turbulence $V^k_{12} \propto \sqrt{k\, k_1\, k_2}$ an integrate over k_2 with the help of the δ-functions. Then the leading term is

$$\begin{aligned}\mathrm{St}_k &\propto \int_0^k \left(\mathcal{N}^k_{k_1,k-k_1} - \mathcal{N}^{k+k_1}_{k,k_1}\right) dk_1 \propto \int_0^k k_1 n_{k_1} \Big[\left(n_{k-k_1} - n_k\right) - \left(n_k - n_{k+k_1}\right)\Big] dk_1 \\ &= \int_0^k k_1 n_{k_1} \Big[n_{k+k_1} + n_{k-k_1} - 2n_k\Big] dk_1 \propto \frac{d^2 n_k}{dk^2} \int_0^k k_1 n_{k_1} k_1^2 dk_1 \propto \int_0^k k_1^3 n_{k_1} dk_1 \ .\end{aligned} \tag{28}$$

We see that this integral converges for any $n_k \propto k^{-x}$ with $x < 4$ including $x = 3$.

D. Ultraviolet locality

Consider now the ultraviolet (UV) locality, i.e. convergence of integral in Eq. (9) of the main text, in the region $k_1 \gg k$. Now the leading term is

$$\begin{aligned}\mathrm{St}_k &\propto \int_k^\infty \left(\mathcal{N}^k_{k_1,k-k_1} - \mathcal{N}^{k+k_1}_{k,k_1}\right) dk_1 \propto \int_k^\infty k_1^2 \Big[n_k n_{k_1-k} - n_{k_1}\left(n_k + n_{k_1-k}\right) + n_k n_{k_1} - n_{k+k_1}\left(n_k + n_{k_1}\right)\Big] dk_1 \\ &\approx n_k \int_k^\infty k_1^2 \Big[n_{k_1-k} - n_{k_1+k}\Big] dk_1 \approx -2k \int_k^\infty k_1^2 \frac{dn_{k_1}}{dk_1} dk_1 \propto \int_k^\infty \frac{k_1^2 dk_1}{k_1^{x+1}} \ .\end{aligned} \tag{29}$$

We see that in the UV-region this integral converges for any $x > 2$, including $x = 3$.

The overall conclusion is that the collision integral converges for $2 < x < 4$ and actual scaling exponent $x = 3$ is exactly in the middle of the locality window. This phenomenon is called counterbalanced locality of the collision integral, which quite common property of the kinetic equations.

III. ENERGY SPECTRUM, ENERGY FLUX, AND CONSTANT C_1

A. Energy spectrum

‡

The total energy (Eq. (19) of the main text) can be rewritten as

$$E = \frac{1}{L^2 \rho_0} \int \left[\frac{\rho}{2}(\nabla\phi)^2 + \frac{c^2}{2\rho_0}(\rho - \rho_0)^2 + c^2\xi^2(\nabla\rho)^2\right] \mathrm{d}^2\boldsymbol{r} = \frac{1}{L^2\rho_0} \int \left[c_\mathrm{s}^2 \xi^2 |\nabla\psi|^2 + \frac{c_\mathrm{s}^2}{2\rho_0}(\rho - \rho_0)^2\right] \mathrm{d}^2\boldsymbol{r}. \tag{30}$$

The energy spectrum is then computed taking into account that the total energy is the sum of two quadratic quantities and using the definition of the cross spectrum of two fields f and g that is defined in terms of their Fourier transform $\hat{f}$ and $\hat{g}$ as

$$E_{f,g}(k) = \frac{1}{\Delta_k} \sum_{k-\Delta_k/2<|\mathbf{k}|<k+\Delta_k/2} \hat{f}_\mathbf{k} \hat{g}^*_\mathbf{k}$$

for some small Δ_k. Note that by the Parseval theorem $\int f(\mathbf{x}) g^*(\mathbf{x}) d\mathbf{x} = L^2 \sum_k E_{f,g}(k)\Delta_k \approx \int_k E_{f,g}(k)\, dk$. The total energy spectrum is then computed as $E(k) = E_\mathrm{kin}(k) + E_\mathrm{int}(k)$, where $\rho_0 E_\mathrm{kin}(k) = c_\mathrm{s}^2 \xi^2 E_{\nabla\psi,\nabla\psi}(k)$ and $2\rho_0^2 E_\mathrm{int}(k) = c_\mathrm{s}^2 E_{\rho-\rho_0,\rho-\rho_0}(k)$.

B. Energy flux

The energy flux can be computed as usual in hydrodynamics [6], but adapting it to GP dynamics as

$$\varepsilon(k) = -\sum_{p=0}^{k} \frac{\partial E(p)}{\partial t}\Big|_{\rm GP} = -\sum_{p=0}^{k} \frac{\partial E_{\rm kin}(p)}{\partial t}\Big|_{\rm GP} + \frac{\partial E_{\rm int}(p)}{\partial t}\Big|_{\rm GP}, \tag{31}$$

where the label GP means that time derivatives are computed using the GP equation (without forcing and dissipation). Namely, we have

$$\frac{\partial E_{\rm kin}(p)}{\partial t}\Big|_{\rm GP} = \frac{2c_{\rm s}^2\xi^2}{\rho_0} E_{\nabla\psi,\nabla d\psi}(k) \tag{32}$$

$$\frac{\partial E_{\rm int}(p)}{\partial t}\Big|_{\rm GP} = \frac{c_{\rm s}^2}{\rho_0^2} E_{\rho-\rho_0,d\rho}(k) \tag{33}$$

where $d\psi = -i\frac{c_{\rm s}}{\sqrt{2}\xi}\left[-\xi^2\nabla^2 + \frac{1}{\rho_0}|\psi|^2 - 1\right]\psi$ and $d\rho = \psi d\psi^* + d\psi\psi^*$. Note that $\lim_{k\to\infty}\varepsilon(k) = 0$ because of the energy conservation of the GP equation.

C. Dimensionless prefactor C_1

To compute C_1, we substitute Eq.(15) into the definition of the flux (6) (both in the main text), substitute $d\boldsymbol{k}_1 = 2\pi k_1 dk_1$ (since the integrand is a function of the $k_1 = |\mathbf{k}_1|$ and the polar angle is immediately integrated out) and integrate with respect to k_1. This leads to

$$\varepsilon_k = -\frac{2\pi^2 A^2 V_0^2}{\sqrt{6}\,a}\frac{I(x)}{(3-x)} k^{6-2x} \ . \tag{34}$$

For actual value $x = 3$ this equation has an uncertainties zero divided by zero, which can be resolved according the the L'Hopital rule:

$$\lim_{x\to 3}\frac{I(x)}{(3-x)} = \frac{dI(x)}{dx}\Big|_{x=3} = \int_0^1 \frac{12\log q_1}{q_1 - 1} dq_1 = 2\pi^2.$$

Now Eq. (34) with $x = 3$ give for the energy flux:

$$\varepsilon_k = \frac{4\pi^4 A^2 V_0^2}{\sqrt{6}\,a} \ . \tag{35}$$

Thus, Eq. (35), together with Eqs.(9) and (11) of the main text finally give:

$$C_1 = \frac{6^{1/4}\sqrt{c_s}}{\pi V_0} \ . \tag{36}$$

for the pre-factor C_1 in Eq. (9) of the main text.

[1] V. E. Zakharov and R. Z. Sagdeev, Dokl. Akad. Nauk SSSR **192**, 297 (1970).
[2] S. Dyachenko, A. C. Newell, A. Pushkarev, and V. E. Zakharov, Physica D: Nonlinear Phenomena **57**, 96 (1992).
[3] V. E. Zakharov and S. V. Nazarenko, Physica D: Nonlinear Phenomena **201**, 203 (2005).
[4] S. Nazarenko, *Wave Turbulence* (Springer Berlin Heidelberg, series: Lecture Notes in Physics, 2011).
[5] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics, Second Edition: Volume 6 (Course of Theoretical Physics)*, Course of theoretical physics / by L. D. Landau and E. M. Lifshitz, Vol. 6 (Butterworth-Heinemann, 1987), 2nd ed., ISBN 0750627670.
[6] U. Frisch, *Turbulence: The Legacy of A.N. Kolmogorov* (Cambridge University Press, 1995), 1st ed., ISBN 978-0-521-45103-1 978-1-139-17066-6 978-0-521-45713-2.