Energy spectrum of two-dimensional acoustic turbulence: Supplemental Material

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I. HAMILTONIAN FORMULATION OF ACOUSTIC TURBULENCE

In this section we review the Hamiltonian formulation of acoustic turbulence and obtain the interaction term represented by $V_{12}^{k}$ in Eq.(5) of the main text. In the acoustic limit, this term is well known to be $V_{12}^{k} = V_{0}\sqrt{k_1k_2}$ [1-4], however the constant $V_{0}$ in these references takes different values. Here, we will carefully derive its value.

The starting point is the action (per unit of mass) for a compressible, isentropic, irrotational fluid [5] that reads:

$$S = \frac{1}{\rho_0L^2} \int dt d^2x \left[ -\rho \frac{\partial}{\partial t}(\rho \phi) - \frac{\rho}{2} (\nabla \phi)^2 - \frac{\sigma^2}{2\rho_0} (\rho - \rho_0)^2 \right],$$  \hspace{1cm} (1)

where $\rho_0$ is the bulk density and $c_0$, as it will be clear later, is the speed of sound. Note that the dimensions of the fields are $[\phi] = L^2/T$ and $[\rho] = M/L^2$.

Varying the action with respect to $\rho$ and $\phi$ and we obtain the fluid equations,

$$\dot{\rho} + \nabla (\rho \nabla \phi) = 0,$$  \hspace{1cm} (2)

$$\dot{\phi} + \frac{1}{2} \nabla \phi^2 = -\frac{c_0^2}{\rho_0} (\rho - \rho_0).$$  \hspace{1cm} (3)

Acoustic waves are readily obtained by linearizing the equations about $\phi = 0$ and $\rho = \rho_0$, which leads to the wave equation $\ddot{\phi} = c_0^2 \nabla^2 \phi$.

Note that the action (1) is not written in a Hamiltonian way. Making the following change of variables

$$\dot{\rho} = \frac{\delta H}{\delta A}, \quad \dot{A} = -\frac{\delta H}{\delta p},$$  \hspace{1cm} (5)

where the equation of motion are now given by

We remark that the units of the new fields are $[A] = 1$ and $[p] = L^2/T$. The Hamiltonian per unit of mass has units $[H] = L^2/T^2$ as usual in hydrodynamics.

A. Acoustic waves

Waves are obtained by making $A \rightarrow 1 + \tilde{A}$ and $p \rightarrow \tilde{p}$. Dropping tildes and keeping the terms up to the cubic order, we rewrite the Hamiltonian as $H = H_2 + H_3$, where the second and third order terms are

$$H_2 = \int \frac{d^2x}{L^2} \left[ \frac{1}{8} (\nabla p)^2 + 2c_0^2 A^2 \right], \quad H_3 = \int \frac{d^2x}{L^2} \left[ 2c_0^2 A^3 - \frac{1}{4} p(\nabla p) \cdot (\nabla A) \right].$$  \hspace{1cm} (6)

We now assume that the fields are periodic and write them as $p(x) = \sum_k p_k e^{ik \cdot x}$ and $A(x) = \sum_k A_k e^{ik \cdot x}$. The Hamiltonian and the action become:

$$S = \int dt \sum_k \frac{1}{2} \left( A_k \dot{p}^2_k - i k \cdot p_k \right) - \int dt(H_2 + H_3),$$  \hspace{1cm} (7)

$$H_2 = \sum_k \frac{1}{8} k^2 |p_k|^2 + 2c_0^2 |A_k|^2,$$  \hspace{1cm} (8)

$$H_3 = \sum_{1,2,3} 2c_0^2 A_1 A_2 A_3 \delta_{1,2,3} + \frac{1}{4} p_1 p_2 A_3 k_2 \cdot k_3 \delta_{1,2,3},$$  \hspace{1cm} (9)
where $\delta_{1,2,3}$ is 1 if $k_1 + k_2 + k_3 = 0$, and 0 otherwise.

In order to write the Hamiltonian and the action in the canonical form, we perform the following change of variables

$$p_k = i \frac{1}{\sqrt{2}} \left( \frac{\alpha}{k^2} \right)^{\frac{1}{2}} (a_k - a^*_k),$$

$$A_k = \frac{1}{\sqrt{2}} \left( \frac{k^2}{\alpha} \right)^{\frac{1}{2}} (a_k + a^*_k),$$

where $\alpha = 16c_8^2$. The value of this coefficient is set in order to kill the off-diagonal terms in $H_2$. At the leading order, the action becomes

$$S_2 = \int dt \frac{i}{2} \sum_k (\dot{a}_k a^*_k - a_k \dot{a}^*_k) - \int dt H_2, \quad \text{with } H_2 = \sum_k c_k k |a_k|^2 = \sum_k \omega_k |a_k|^2. \quad (12)$$

Then,

$$\frac{\delta S_2}{\delta a^*_k} = 0 \implies i \dot{a}_k = \frac{\partial H}{\partial a_k^*} = \omega_k a_k. \quad (13)$$

**B. $H_3$ terms**

The cubic part of the Hamiltonian requires some tedious work. Keeping only resonant terms we obtain the following contributions

$$\sum A_1 A_2 A_3 \delta_{1,2,3} = \frac{1}{2^{3/2}} \sum_{1,2,3} \frac{\sqrt{k_1 k_2 k_3}}{\alpha^3} (a_1 + a^*_1)(a_2 + a^*_2)(a_3 + a^*_3) = 3 \frac{2^{3/2}}{2^{3/2}} \sum \frac{\sqrt{k_1 k_2 k_3}}{\alpha} (a_1 a_2 a_3^* + a_1^* a_2 a_3) \delta_{1,2,3}^1, \quad (14)$$

where $\delta_{1,2,3}^1 = \delta_{-1,2,3}$. The second term requires more manipulations

$$\sum p_1 p_2 A_3 \cdot k_3 \delta_{1,2,3} = - \frac{1}{2^{3/2}} \sum_{1,2,3} \frac{\alpha^3}{\sqrt{k_1 k_2 k_3}} k_3 (k_2 \cdot k_3)(a_1 - a^*_1)(a_2 - a^*_2)(a_3 + a^*_3) \delta_{1,2,3}$$

$$= - \frac{1}{2 \times 2^{3/2}} \sum_{1,2,3} \frac{\alpha^3}{\sqrt{k_1 k_2 k_3}} k_3 (k_1 \cdot k_3 + k_2 \cdot k_3)(a_1 - a^*_1)(a_2 - a^*_2)(a_3 + a^*_3) \delta_{1,2,3}$$

$$= \frac{1}{2 \times 2^{3/2}} \sum_{1,2,3} \frac{\alpha^3}{\sqrt{k_1 k_2 k_3}} k_3^2 (a_1 - a^*_1)(a_2 - a^*_2)(a_3 + a^*_3) \delta_{1,2,3}, \quad (15)$$

where form the second to third line we used the resonant condition $k_1 + k_2 = -k_3$. Again, keeping only resonant terms, changing summation variables and using symmetries, we can replace inside the sum

$$k_3^2 (a_1 - a^*_1)(a_2 - a^*_2)(a_3 + a^*_3) \delta_{1,2,3} \rightarrow (a_1 a_2 a_3 + a_1^* a_2 a_3) (k_1^3 - 2k_2^3) \delta_{1,2,3} \rightarrow (a_1 a_2 a_3 + a_1^* a_2 a_3) (k_1^3 - k_2^3 - k_3^3) \delta_{1,2,3} \quad (18)$$

Finally, using the resonant condition $k_1 = k_2 + k_3$, we have $k_1^3 - k_2^3 - k_3^3 = (k_2 + k_3)^3 - k_2^3 - k_3^3 = 3(k_2 + k_3)k_2 k_3 = 3k_1 k_2 k_3$. Gathering all the terms

$$H_3 = \frac{1}{2^{3/2}} \sum (a_1 a_2 a_3^* + c.c) \alpha_1 \sqrt{k_1 k_2 k_3} \left[ 2c_8^2 \frac{3}{\alpha^{3/4}} + \frac{1}{3} \frac{3\alpha^{1/4}}{4} \right] = \sum V_{23}^1 (a_1 a_2 a_3^* + c.c) \delta_{1,2,3}^1, \quad (19)$$

where $V_{23}^1 = V_0 \sqrt{k_1 k_2 k_3}$, with

$$V_0 = \frac{1}{2^{3/2}} \left[ 2c_8^2 \frac{3}{\alpha^{3/4}} + \frac{1}{3} \frac{3\alpha^{1/4}}{4} \right] = \frac{3}{4} \sqrt{c_8}, \quad (20)$$

the formula used to obtained Eq. (18) from Eq. (16) in the main text.
FIG. 1.
Wave vector triad. We choose a coordinate system such that \( \vec{k} \) is aligned with the \( x \)-axis. \( \theta_1 \) is the angle between \( \vec{k} \) and \( \vec{k}_1 \), \( k_{1y} \) is the \( y \)-component of \( \vec{k}_1 \).

II. ANALYSIS OF THE COLLISION TERM

A. Analysis of \( \mathcal{R}_{12}^k \) term

According to Eqs. (3) of the main text, the first term in the collision integral (\( \mathcal{R}_{12}^k \)) contains

\[
\Delta_{12}^k \equiv \int_{k_{1x}, k_{1y}} \delta(\omega_{12}) \delta_{12}^k \, dk_1 \, dk_2 \approx \frac{k_1 k_2 \, dk_{1x}}{c_s \, k |k_{1y}|}.
\]

From Fig. 1 we can see that to leading order

\[
k_{1y} = k_1 \sin(\theta_1 k) \approx k_1 \theta_1 k.
\]

To find \( \theta_1 k \) consider the wave number resonance condition,

\[
k_2 = k - k_1 \Rightarrow k_2^2 = k_1^2 + k^2 - 2k_1 \cdot k = k_1^2 + k^2 - 2k_1 k \cos \theta_1 k \approx (k - k_1)^2 + k k_1 \theta_1^2 k
\]

\[
\Rightarrow k_2 \approx (k - k_1) + \frac{k k_1}{2k_2} \theta_1^2 k .
\]

Next, consider the frequency resonance condition,

\[
k_1 + k_2 - k = -a^2(k_1^3 + k_2^3 - k^3),
\]

and substitute \( k_2 \) from Eq. (23) to the LHS and \( k_2 = (k - k_1) \) to the RHS of this equation. This gives \( \frac{k k_1 \theta_1^2 k}{2k_2} = 3a^2 k k_1 k_2 \) or \( \theta_1^2 k_1 = 6a^2 k_2^2 \) and

\[
k_{1y} = \sqrt{6a} k_1 k_2.
\]

Together with Eq. (21), this finally gives

\[
\Delta_{12}^k \approx \frac{dk_{1x}}{\sqrt{6} c_s \, a k} = \frac{\delta(k - k_1 - k_2) \, dk_1 \, dk_2}{\sqrt{6} c_s \, a k},
\]

also shown in Eq. (8.b) in the main text. Here, we have replaced \( dk_{1x} \) by \( dk_1 \) and inserted \( \delta(k - k_1 - k_2) \, dk_2 = 1 \) to stress that \( k = k_1 + k_2 \) in the used approximation.

B. Contribution to \( \mathcal{R}_{k2}^1 \) and \( \mathcal{R}_{k1}^2 \)

Similar derivations using the wave number and frequency resonance conditions lead to

\[
\Delta_{2k}^1 \approx \frac{dk_{1x}}{\sqrt{6} c_s \, a k} = \frac{\delta(k_1 - k - k_2) \, dk_1 \, dk_2}{\sqrt{6} c_s \, a k} \quad \text{and} \quad \Delta_{1k}^2 \approx \frac{\delta(k_2 - k - k_1) \, dk_1 \, dk_2}{\sqrt{6} c_s \, a k} .
\]

Together, Eqs. (26) and (27) lead to the collision integral (10) of the main text.
C. Proof of the interaction locality

Convergence of the integral in Eq. (10) of the main text is referred to as the interaction locality property. First, we note that this integral is trivially convergent for $x = 1$ because the integrand is identically equal to zero. This exponent corresponds to the thermodynamic energy equipartition state, i.e. a trivial zero-flux equilibrium which we will not be interested in. Thus, below we will consider the cases with $x \neq 1$.

1. Infrared locality

Consider first the infra-red (IR) locality, i.e. convergence of the integral in Eq. (10) of the main text, in the region $k_1 \ll k$. We take into account that for the acoustic turbulence $V_{12}^k \propto \sqrt{k} k_1 k_2$ an integrate over $k_2$ with the help of the $\delta$-functions. Then the leading term is

$$\text{St}_k \propto \int_0^k \left( N_{k_1,k-k_1}^k - N_{k_1,k_1}^{k+k_1} \right) dk_1 \propto \int_0^k k_1 n_{k_1} \left[ (n_{k-k_1} - n_k) - (n_k - n_{k+k_1}) \right] dk_1$$

$$= \int_0^k k_1 n_{k_1} \left[ n_{k-k_1} - n_k \right] dk_1 \propto \frac{d^2 n_k}{dk^2} \int_0^k k_1 n_{k_1} k_1^2 dk_1 \propto \int_0^k k_1^3 n_{k_1} dk_1 .$$

We see that this integral converges for any $n_k \propto k^{-x}$ with $x < 4$ including $x = 3$.

D. Ultraviolet locality

Consider now the ultraviolet (UV) locality, i.e. convergence of integral in Eq. (9) of the main text, in the region $k_1 \gg k$. Now the leading term is

$$\text{St}_k \propto \int_k^\infty \left( N_{k_1,k-k_1}^k - N_{k_1,k_1}^{k+k_1} \right) dk_1 \propto \int_k^\infty k_1^2 \left[ n_{k-k_1} - n_k \right] \left( n_k + n_{k_1} \right) n_{k+k_1} \left( n_k + n_{k_1} \right) dk_1$$

$$\simeq n_k \int_k^\infty k_1^2 \left[ n_{k-k_1} - n_{k+k_1} \right] dk_1 \propto -2k \int_k^\infty k_1^2 \frac{dn_k}{dk_1} dk_1 \propto \int_k^\infty k_1^4 dk_1 .$$

We see that in the UV-region this integral converges for any $x > 2$, including $x = 3$.

The overall conclusion is that the collision integral converges for $2 < x < 4$ and actual scaling exponent $x = 3$ is exactly in the middle of the locality window. This phenomenon is called counterbalanced locality of the collision integral, which quite common property of the kinetic equations.

III. ENERGY SPECTRUM, ENERGY FLUX, AND CONSTANT $C_1$

A. Energy spectrum

The total energy (Eq. (19) of the main text) can be rewritten as

$$E = \frac{1}{L^2 \rho_0} \int \left[ \frac{\rho}{2} (\nabla \phi)^2 + \frac{c^2}{2\rho_0} (\rho - \rho_0)^2 + c^2 \xi^2 (\nabla \psi)^2 \right] d^2 \mathbf{r} = \frac{1}{L^2 \rho_0} \int \left[ c^2 \xi^2 |\nabla \psi|^2 + \frac{c^2}{2\rho_0} (\rho - \rho_0)^2 \right] d^2 \mathbf{r} .$$

The energy spectrum is then computed taking into account that the total energy is the sum of two quadratic quantities and using the definition of the cross spectrum of two fields $\tilde{f}$ and $\tilde{g}$ that is defined in terms of their Fourier transform $\tilde{f}$ and $\tilde{g}$ as

$$E_{f,g}(k) = \frac{1}{\Delta_k} \sum_{k-\Delta_k/2 < |k| < k+\Delta_k/2} \tilde{f}_k \tilde{g}_k^*$$

for some small $\Delta_k$. Note that by the Parseval theorem $\int f(x)g^*(x)dx = L^2 \sum_k E_{f,g}(k) \Delta_k \approx \int_k E_{f,g}(k) dk$. The total energy spectrum is then computed as $E(k) = E_{\text{kin}}(k) + E_{\text{int}}(k)$, where $\rho_0 E_{\text{kin}}(k) = c^2 \xi^2 E_{\nabla \psi, \nabla \psi}(k)$ and $2 \rho_0^2 E_{\text{int}}(k) = c^2 E_{\rho - \rho_0, \rho - \rho_0}(k)$.
B. Energy flux

The energy flux can be computed as usual in hydrodynamics [5], but adapting it to GP dynamics as

\[ \varepsilon(k) = -\sum_{p=0}^{k} \frac{\partial E(p)}{\partial t} \bigg|_{\text{GP}} = -\sum_{p=0}^{k} \frac{\partial E_{\text{kin}}(p)}{\partial t} \bigg|_{\text{GP}} + \frac{\partial E_{\text{int}}(p)}{\partial t} \bigg|_{\text{GP}}, \]  

(31)

where the label GP means that time derivatives are computed using the GP equation (without forcing and dissipation). Namely, we have

\[ \frac{\partial E_{\text{kin}}(p)}{\partial t} \bigg|_{\text{GP}} = 2c^2 s \xi^2 \rho_0 E_k \nabla^2 \psi \nabla \psi(k) \]  

(32)

\[ \frac{\partial E_{\text{int}}(p)}{\partial t} \bigg|_{\text{GP}} = c^2 \rho_0 \left( E_{\rho^2} - \rho \frac{d\rho}{dt} \right) \psi(k) \]  

(33)

where \( d\psi = -i \frac{\xi}{\sqrt{2}} \left[ -\xi^2 \nabla^2 + \frac{1}{\rho} |\psi|^2 - 1 \right] \psi \) and \( d\rho = \psi d\psi^* + d\psi^* \psi \). Note that \( \lim_{k \to \infty} \varepsilon(k) = 0 \) because of the energy conservation of the GP equation.

C. Dimensionless prefactor \( C_1 \)

To compute \( C_1 \), we substitute Eq.(15) into the definition of the flux (6) (both in the main text), substitute \( dk_1 = 2\pi k_1 dk_1 \) (since the integrand is a function of the \( k_1 = |k_1| \) and the polar angle is immediately integrated out) and integrate with respect to \( k_1 \). This leads to

\[ \varepsilon_k = -\frac{2\pi^2 A^2 V_0^2}{\sqrt{6}a} \frac{I(x)}{(3-x)} k_1^{6-2x} \]  

(34)

For actual value \( x = 3 \) this equation has an uncertainties zero divided by zero, which can be resolved according the the L’Hopital rule:

\[ \lim_{x \to 3} \frac{I(x)}{(3-x)} = \frac{dI(x)}{dx} \bigg|_{x=3} = \int_0^1 \frac{12 \log q_1 dq_1}{q_1 - 1} = 2\pi^2. \]

Now Eq. (34) with \( x = 3 \) give for the energy flux:

\[ \varepsilon_k = \frac{4\pi^4 A^2 V_0^2}{\sqrt{6}a}. \]  

(35)

Thus, Eq. (35), together with Eqs.(9) and (11) of the main text finally give:

\[ C_1 = \frac{6^{1/4} \sqrt{\frac{\varepsilon}{\pi V_0}}}{\pi V_0}. \]  

(36)

for the pre-factor \( C_1 \) in Eq. (9) of the main text.