Comment on “Superfluid Turbulence from Quantum Kelvin Wave to Classical Kolmogorov Cascades”

In a recent Letter [1], Yepez et al. performed numerical simulations of the Gross-Pitaevskii equation (GPE) using a novel unitary quantum algorithm with very high resolution. They claim to have found new power-law scalings for the incompressible kinetic energy spectrum: “… (the) solution clearly exhibits three power law regions for $E_{\text{kin}}^{\text{comp}}(k)$: for small $k$ the Kolmogorov $k^{-(5/3)}$ spectrum while for high $k$ a Kelvin wave spectrum of $k^{-3}$ …”

In this Comment we point out that the high wave number $k^{-3}$ power law observed by Yepez et al. is an artifact stemming from the definition of the kinetic energy spectra and is thus not directly related to a Kelvin wave cascade. Furthermore, we clarify a confusion about the wave number intervals on which Kolmogorov and Kelvin wave cascades are expected to take place.

The dynamics of a superfluid is described by the GPE

$$\partial_t \psi = i c / (\sqrt{\xi}) (\psi - |\psi|^2 \psi + \xi^2 \nabla^2 \psi),$$  \hspace{1cm} (1)

where the complex field $\psi$ is related by Madelung’s transformation $\psi = \sqrt{\rho} \exp(i \phi / \sqrt{\rho})$ to the density $\rho$ and velocity $\vec{v} = \nabla \phi$ of the superfluid. In these formulas, $\xi$ is the coherence length and $c$ is the velocity of sound (for a fluid of unit mean density). The superflow is irrotational, except on the nodal lines $\psi = 0$, which are the superfluid vortices.

The GPE dynamics Eq. (1) conserves the energy that can be written as the sum (the space integral) of three parts: the kinetic energy $E_{\text{kin}} = 1/2 (\sqrt{\rho} \vec{v})^2$, the internal energy $E_{\text{int}} = (c^2/2)(\rho - 1)^2$, and the quantum energy $E_q = c^2 \xi^2 (\partial / \sqrt{\rho})^2$. Using Parseval’s theorem, one can define the corresponding energy spectra, e.g., the kinetic energy spectrum $E_{\text{kin}}(k)$, as the sum over the angles of $\int d^3 \rho e^{i k \cdot \rho} (\sqrt{\rho} \vec{v})^2$ [2].

The 3D angle-averaged spectrum of a smooth isolated vortex line is known to be proportional to that of the 2D axisymmetric vortex, an exact solution of Eq. (1) given by $\psi_{\text{vort}}(r) = \sqrt{\rho(r)} \exp(\pm i \varphi)$ in polar coordinates $(r, \varphi)$. The corresponding velocity field $v(r) = \sqrt{2c\xi/r}$ is azimuthal and the density profile, of characteristic spatial extent $\xi$, verifies $\sqrt{\rho(r)} \sim r$ as $r \to 0$ and $\sqrt{\rho(r)} = 1 + O(r^{-2})$ for $r \to \infty$. Thus $\sqrt{\rho} v_j$ has a small $r$ singular behavior of the type $\rho^3$ and behaves as $r^{-1}$ at large $r$. In general, for a function scaling as $g(r) \sim r^l$ the (2D) Fourier transform is $\tilde{g}(k) \sim k^{-l-2}$ and the associated spectrum scales as $k^{-2l-3}$. Thus $E_{\text{kin}}(k)$ scales as $k^{-3}$ for $k \gg k_\xi \sim \xi^{-1}$ and as $k^{-1}$ for $k \ll k_\xi$ [3].

Following the above discussion, the $k^{-3}$ power law observed in [1] is an artifact stemming from the definition of the kinetic energy spectra and is not directly related to a Kelvin wave cascade.

Another very important scale, not discussed in the Letter [1], is the scale $\ell$ of the mean intervortex distance. The hydrodynamic (Kolmogorov) energy cascade is expected to end at $k_\ell \sim \ell^{-1}$ [2] and the Kelvin wave cascade to begin there, after an eventual bottleneck [4]. Note that $\ell / \xi \gg \ell / \ell_j$, where $\ell_j$ is the energy containing scale. We thus believe that nothing particularly interesting is taking place between $k_\xi$ and the maximum wave number $k_{\text{max}}$ of the simulation and that there is a confusion in [1] between $k_\ell$ and $k_{\text{max}}$.

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