

Wave-Kinetic Dynamics of Forced-Dissipated Turbulent Internal Gravity Waves

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Internal gravity waves are an essential feature of stratified media, such as oceans and atmospheres. To investigate their dynamics, we perform simulations of the forced-dissipated kinetic equation describing the evolution of the energy spectrum of weakly nonlinear internal gravity waves. During the early evolution, three well-known nonlocal interactions, the elastic scattering, the induced diffusion, and the parametric subharmonic instability, together with a superharmonic resonance play a prominent role. In contrast, local interactions are responsible for anisotropic energy cascade on longer timescales. We reveal emergence of a condensate at small horizontal wave vectors that can be interpreted as a pure wave-wave interaction-mediated layering process.

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When a stable stratification gradient of salinity or temperature is present in a fluid, a peculiar type of wave can propagate in its bulk [1]. Such perturbations are called internal gravity waves (IGWs), and unlike most common waves, their wavelengths and wave frequencies are not monotonically related; the IGW dispersion relation is given by $\omega_{\mathbf{k}} = Nk_h/|\mathbf{k}|$, where \mathbf{k} is the wave vector, k_h its horizontal magnitude (in the plane perpendicular to gravity), and N the Brunt-Väisälä frequency [1]. The inherently anisotropic character of the IGWs makes their dynamics rich and complex, even at the linear level [2,3]. The IGWs also play a crucial role in ocean mixing [4–7]. One of the first descriptions of the ocean internal wave field was given by Garrett and Munk (GM) [8,9], who furnished an empirical expression for the wave energy spectra. Since then, many theoretical works have followed GM phenomenology to explain the mechanisms behind the formation of such spectra. From a more theoretical side, the weak wave turbulence (WWT) theory [10–13] provides a wave kinetic equation (WKE) describing the evolution of IGWs under the effect of weakly nonlinear interactions. This equation has been derived using various approaches and approximations; see, e.g., [14–19]. It always takes the form $\dot{n}_{\mathbf{k}} = St_{\mathbf{k}}[n_{\mathbf{k}}]$, where \mathbf{k} is the wave vector and $n_{\mathbf{k}}$ is the 3D wave action spectrum, which is proportional to the square of the magnitude of the Fourier transform of the wave amplitude. $St_{\mathbf{k}}$ is the collision integral describing interactions between waves. The WWT is a general theory that not only applies to IGWs, but to many other systems such as plasma waves [20], gravitational waves [21], turbulence in Bose-Einstein condensates [22,23], and so

on [13]. The theory typically furnishes analytical predictions that can be tested in experiments and numerical simulations.

The WKE gives access to the wave dynamics on kinetic (very large) timescales that cannot be easily accessed through direct numerical simulations of the fluid dynamical equation in the weakly nonlinear regime. In many fields, numerical simulations of WKEs have become a powerful numerical tool to describe nonlinear wave cascades and transient states [23–27]. For example, WKE simulations are central to the so-called “wave models” used for sea state forecasts [28]. In the case of IGWs, simulations of the WKE are delicate and have a cost \propto (No. of grid points per dimension)⁴, which has challenged the study of steady-state turbulent IGW. For this reason, early numerical IGW works in the 70s focused on the estimation of energy transfers by computing the IGW collision integral for some test spectra (particularly for GM) [4,14,29–31]. Those studies revealed that nonlocal interactions (i.e., involving modes with very different wave-vector lengths or wave frequencies) are essential. Namely, three mechanisms were identified by McComas and Bretherton [29]: (i) *elastic scattering* (ES), in which two waves of similar vertical wave number modulus $k_{2z} \simeq -k_{1z}$ interact with another one having a lower frequency and twice the vertical wave number $k_z = k_{2z} - k_{1z} \simeq 2k_{2z}$; (ii) *parametric subharmonic instability* (PSI), in which a low-wave-number mode decays into two with high wave numbers with half the frequency, i.e., $\omega_{\mathbf{k}}/2 \simeq \omega_1 \simeq \omega_2$; and (iii) *induced diffusion* (ID), in which one low-frequency wave interacts with two approximately identical waves of much larger wave number and frequency. As shown in [32], ID proceeds along lines $\xi_{\mathbf{k}} \equiv k_h/k_z^2 = k_{1h}/k_{1z}^2 = k_{2h}/k_{2z}^2 = \text{const}$, similar to the case of nonlocal drift waves [33].

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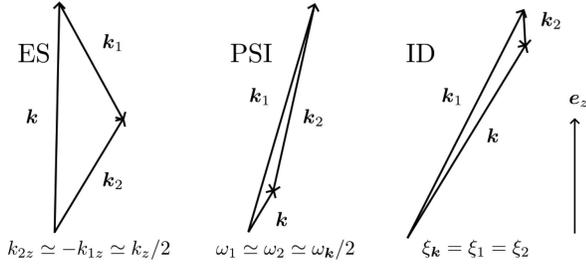


FIG. 1. A cartoon of the three nonlocal interactions identified by McComas and Bretherton [29]. e_z is the vertical axis.

A sketch of the three nonlocal interactions is displayed in Fig. 1. Moreover, a fourth nonlocal interaction is also relevant [34,35]: *superharmonic resonance* (SR), in which two waves having similar frequencies $\omega_1 \simeq \omega_2$ generate a third wave with twice frequency $\omega_k \simeq 2\omega_1$.

More recent studies revisited the energy transfers for specific spectra, and showed that both, local and nonlocal interactions are important [36–38]. Therefore, all interactions should be considered when dealing with the IGW evolution, which makes the analytical description of such a system difficult.

The WWT predicts nonequilibrium, scale-invariant steady-state spectra [15,16,39] that, in the hydrostatic limit, are bihomogeneous in horizontal and vertical wave-vector amplitudes k_h and $|k_z|$ (i.e., spectra $\propto k_h^{\nu_h} |k_z|^{\nu_z}$). However, a rigorous analysis showed that most of these spectra are not mathematically valid solutions because the collisional integral diverges [40]. The only exception is a spectrum found numerically on the line $\nu_z = 0$, which corresponds to the exact cancellation of two divergent parts of the collision integral [40,41]. Note that for this solution, any perturbation can lead to a divergence of the collision integral, which puts the physical realizability of this spectrum in question. Finally, considering only nonlocal ID interactions yields yet another analytical prediction [32].

Current theoretical predictions are all limited to steady-state spectra and are hardly realizable in experiments or direct numerical simulations of the Navier-Stokes-Boussinesq equations. In this Letter, we present numerical simulations of the forced-dissipated WKE for IGWs in the hydrostatic limit, i.e. $k_h \ll |k_z|$. The hydrostatic approximation, commonly used in oceanographic theoretical studies [7,36,40], allows for great simplifications as the resonant manifold (defined later) can be treated analytically and the WKE becomes bihomogeneous in k_h and $|k_z|$. The latter property allows us to place a localized forcing in any sector of our computational box to explore different regimes. Our simulations allow us to observe the evolution of the energy spectrum due to weakly nonlinear wave-wave interactions and the establishment of a nontrivial steady-state spectrum.

We consider the WKE derived by Lvov and Tabak [16], to which we add a forcing and a dissipative term,

$$\dot{n}_k = St_k + \mathcal{F}_k - \mathcal{D}_k n_k \quad (1)$$

$$St_k = \int [\mathcal{R}_{12}^k - \mathcal{R}_{k2}^1 - \mathcal{R}_{k1}^2] d^3k_1 d^3k_2 \quad (2)$$

$$\mathcal{R}_{12}^k = 4\pi\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\delta(\omega_k - \omega_1 - \omega_2) \times |V_{k12}|^2(n_1 n_2 - n_k n_1 - n_k n_2), \quad (3)$$

where $\omega_k = Nk_h/|k_z|$ is the hydrostatic wave frequency and V_{k12} is the interaction coefficient; see Supplemental Material (SM) for details [42]. Waves interact only if their corresponding wave vectors lay on the resonant manifold defined by the equations

$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2, \quad \omega_k = \omega_1 + \omega_2, \quad (4)$$

and the respective equation with cyclic permutation of indexes.

We consider axisymmetric and vertically symmetric spectra, i.e., depending only on $(k_h, |k_z|)$, such that the 2D wave energy spectrum is defined as

$$e(k_h, k_z) = 4\pi k_h \omega_k n_k. \quad (5)$$

Note that with this definition, the total energy per unit of volume is $E = \int_0^\infty \int_0^\infty e(k_h, k_z) dk_h dk_z$. The dissipation coefficient \mathcal{D}_k in (1) is defined such that energy is efficiently dissipated for $k_h, |k_z| \lesssim k_{\text{inf}}$ and $\sqrt{k_h^2 + k_z^2} \gtrsim k_{\text{sup}}$. In this Letter, we consider a localized forcing \mathcal{F}_k centered at (k_{fh}, k_{fz}) , so that the rate of the energy injection is $P = 4\pi \int_0^\infty \int_0^\infty \omega_k \mathcal{F}_k k_h dk_h dk_z$. We express time in units of the nonlinear timescale $\tau_{nl} = (k_f^2 P / \omega_f)^{-1/2}$, with $1/k_f$ being the typical forcing length scale and $\omega_f = Nk_{fh}/k_{fz}$ is the typical frequency of the forced waves. We use a logarithmic grid of $M_h \times M_z$ points spanning the spectral domain $[k_{h,\text{min}}; k_{h,\text{max}}] \times [k_{z,\text{min}}; k_{z,\text{max}}]$. We fix the dissipative scales k_{inf} and k_{sup} close to the boundaries of the k domain to enlarge the inertial range but ensuring a well resolved spectrum. For each grid point, the collision integral is given by a 2D integral, obtained after the parametrization of the resonant manifold and which is computed using the trapezoidal rule on the log grid. The numerical cost per time step is then of the order of $M_h^3 M_z$ operations. The collisional integral has integrable divergences that are treated analytically. Intermesh values are obtained using a bilinear interpolation scheme, which preserves positivity. Time advancement is done by the Runge-Kutta 2 method, with an adaptive time step based on the timescale of the collision integral. See SM [42] for more details on the WKE, its resonant manifold and on the numerical method. In this Letter, we present a short simulation spanning a large spectral domain, and a longer one on a reduced domain using $M_h = M_z = 80$ grid points in both simulations. Moreover, to emphasize the

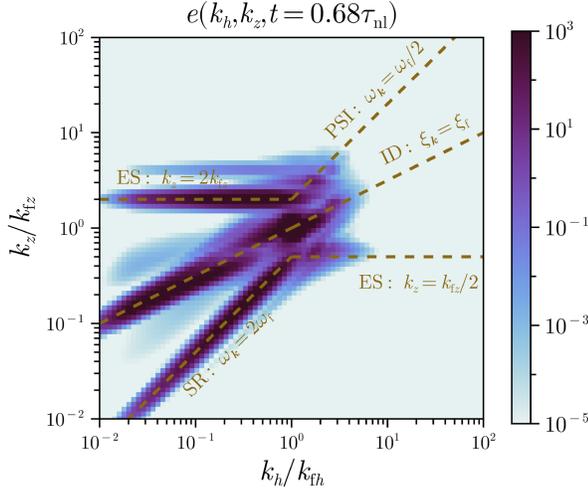


FIG. 2. Energy spectrum $e(k_h, k_z, t)$ in the transient state for a simulation with a localized forcing in the middle of the $(k_h, |k_z|)$ space. The dashed lines indicate wave vectors corresponding to nonlocal interactions with the forced wave vectors $(k_{fh}, |k_{fz}|)$: $k_z = 2k_{fz}$ and $k_z = k_{fz}/2$ (ES), $\omega_k = \omega_f/2$ (PSI), $\xi_k = \xi_f$ (ID), and $\omega_k = 2\omega_f$ (SR).

robustness of our results, SM [42] includes some simulations with different types of forcings and initial conditions.

We first run a simulation with forcing located in the middle of the (k_h, k_z) space, which spans four decades in each direction. Initially, the energy spectrum is zero everywhere, and then it quickly develops along three main lines in the $k_h, k_z < k_f$ sector, as shown in Fig. 2 at $t = 0.68\tau_{nl}$. Evolution's timescale is about a nonlinear time. The horizontal lines correspond to the ES, with $k_{2z} = -k_{1z} = k_{fz}$ so that $k_z = 2k_{fz}$. ID proceeds along the line $\xi = \xi_f = k_{fh}/k_{fz}^2$ as shown in [32], similarly to drift waves [33]. Finally, the third line corresponds to the SR with the forcing at $\omega_k = 2\omega_f$. Note that these excited branches are responsible for an energy transfer toward large scales. Such an inverse energy transfer is consistent with the fact that IGWs are known to concentrate energy at large horizontal scales [44–48]. Here, such an inverse energy transfer is the result of the resonant three-wave interactions only, as observed in [18]. We also notice the development of the spectrum along three other lines, toward small scales $k_h, k_z > k_f$. The first line is $k_z = k_{fz}/2$ and is explained by the ES triadic interactions [37]. The second line corresponds to the PSI with the forcing at $\omega_k = \omega_f/2$. The third line is due to the ID.

Propagation of the spectrum toward higher wave vectors, including the PSI, is much slower and computationally demanding. In this Letter, we will focus on the inverse energy transfers in the (k_h, k_z) space, which have an important impact on the IGWs dynamics and have received less attention than the other type of transfers [49]. Consequently, we now set the forcing at high wave vectors

and run a new simulation in a smaller box until the system reaches a quasisteady state. We show the evolution of the energy spectrum in Fig. 3. The colored curves in panels (c) and (d) show the trajectory of the integral scales

$$\left(\frac{1}{K_h(t)}, \frac{1}{K_z(t)} \right) = \frac{1}{E} \int \left(\frac{1}{k_h}, \frac{1}{k_z} \right) e(k_h, k_z) dk_h dk_z \quad (6)$$

in temporal interval $[0, t]$.

Panel (a) displays a very early time, with almost all the energy being contained in the forced modes. Panel (b) shows the same interactions as in Fig. 2. Panel (c), taken at five nonlinear times, shows that energy starts spreading from the branches excited by the nonlocal interactions, in a process resembling a local energy cascade, but completely anisotropic. We also observe that the integral scale of the flow has moved away from the forcing region. Finally, panel (d) displays the energy spectrum at the end of our simulations, when the system is very close to a steady state. We observe that the energy can be transferred to higher frequencies (i.e., $k_h \gg k_z$), which is consistent with other studies [37,41]. Moreover, nonlocal interactions have transferred most of the energy toward large scales and small frequencies, and local interactions have smoothed out the spectrum so that the forcing region is hardly visible. Interestingly, a new peak emerges at low k_h and high k_z that overwhelms the energy spectrum, as manifested by the evolution of integral scale that has stopped close to this scale. Such a peak resembles a condensate and its formation could be interpreted as a layering process, which is often observed in experiments and simulations [50]. Layering can be seen as the condensation of energy at the lowest wave frequency for stratified flows. In the presence of rotation, the condensation will be saturated at the rotation frequency. Remarkably, the emergence of layers is produced by nonlinear wave interactions only, as shear modes and the vertical vorticity are not taken into account in the wave-kinetic description of IGWs.

The state observed in Fig. 3(d) corresponds to a non-equilibrium turbulent solution of the WKE (1), in which a localized energy input by the forcing balances the dissipation acting at well-separated scales. Turbulent solutions are often characterized by the emergence of power-law spectra in the inertial range, i.e., in between (and far from) the forcing and the dissipation scales. IGWs are highly anisotropic, but their WKE collision integral kernel is bihomogenous. It is then natural to look for solutions of the form $n_k \propto k_h^{\nu_h} |k_z|^{\nu_z}$. Moreover, due to the properties of the ES, the PSI, and the ID, it is also natural to consider solutions in the (ω_k, k_z) plane, as is typically done in oceanographic studies [7]. A simple algebra, and a change of variable, gives for the energy spectrum [51]

$$e(k_h, k_z) \propto k_h^{\nu_h+2} k_z^{\nu_z-1}, \quad \text{and} \quad e(\omega_k, k_z) \propto \omega_k^{\nu_h+2} k_z^{\nu_h+\nu_z+2}. \quad (7)$$

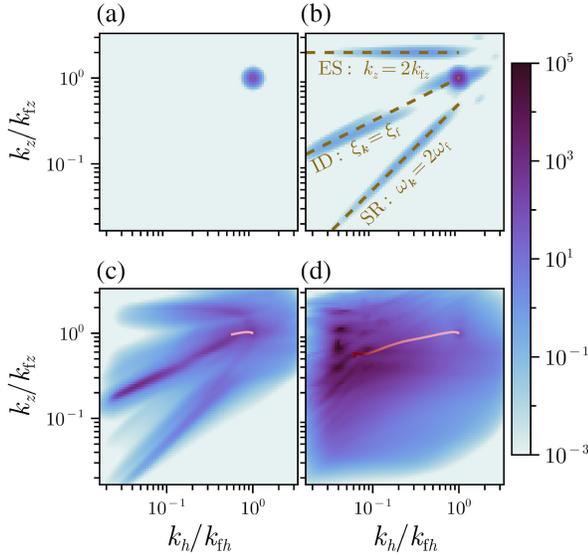


FIG. 3. Snapshots of the energy spectrum $e(k_h, k_z, t)$ at different times for the simulation with a small-scale forcing. On the panel (b), dashed lines are for $k_z = 2k_{fz}$, $\omega_k = 2\omega_f$, and $\xi_k = \xi_f$, corresponding, respectively, to ES, SR, and ID. The colored line represents the trajectory $[K_h(t), K_z(t)]$, defined in equation (6), with dark red colors corresponding to larger t . (a) $t = 0.1\tau_{nl}$, (b) $t = 1.0\tau_{nl}$, (c) $t = 5.0\tau_{nl}$, (d) $t = 580.01\tau_{nl}$.

Some values of (ν_h, ν_z) can be found in the literature, which we discuss in the following. The high-frequency limit of the GM spectrum has the exponents $(-4, 0)$ [8], whereas the Rayleigh-Jeans thermal equilibrium spectrum corresponds to $(-1, 1)$ [15]. The steady state spectrum obtained numerically by Lvov *et al.* [40] corresponds to $(-3.69, 0)$ and the nonlocal ID spectrum of Lanchon and Cortet [32] to $(-3, -1)$.

Figures 4(a) and 4(b), show slices of the 2D energy spectrum shown in Fig. 3(d) for fixed values of, respectively, $k_h = k_{fh}$ and $k_z = k_{fz}$ at different times. In Fig. 4(a), we clearly observe the early stage where energy increases at the forcing scale, followed by energy transfer to larger scales, akin to an inverse cascade with a spectral front propagation. In the figure, we display the GM (k_h^{-2}) and the Lvov *et al.* [40] ($k_h^{-1.69}$) spectral slopes for comparison. Although both scalings seem to be in reasonable agreement with numerics, one has to be careful with such a comparison because the full 2D spectrum deviates from the bihomogenous shape implied by both GM and Lvov *et al.* [40] spectra. At the latest times, a condensate peak is clearly visible at a small k_h . The evolution of the spectrum for a fixed k_h , shown in Fig. 4(b), is more intricate. At wave vectors larger than k_{fz} , a spectrum steeper than k_z^{-1} , expected for GM and the prediction of [40], is observed.

Finally, in Fig. 5 we display cuts of $e(\omega_k, k_z)$ at the latest time of our simulation. Figures 5(a) and 5(b) show the raw data, whereas in Figs. 5(c) and 5(d) the spectra are compensated by the apparent scalings. When the spectrum

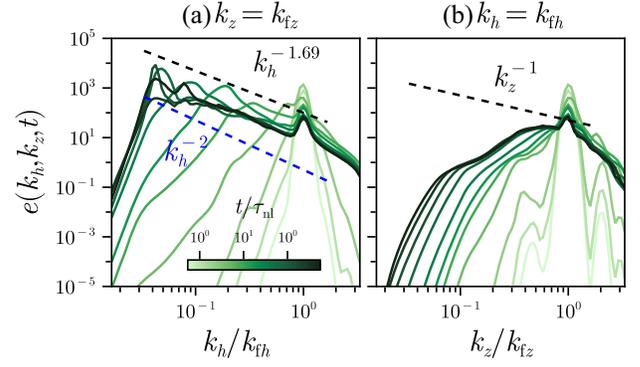


FIG. 4. Slices of the energy spectrum at different times: (a) $e(k_h, k_z = k_{fz}, t)$ and (b) $e(k_h = k_{fh}, k_z, t)$.

is compensated by $\omega_k^{-1.69}$, a good collapse of the curves is obtained as shown in Fig. 5(c). The $\omega_k^{-1.69}$ scaling is also visible in Fig. 5(d), which has been compensated by the k_z^{-4} scaling observed for $k_z > k_{fz}$. To our knowledge, such a steep k_z dependence has not been reported before. However, it could result from the limited inertial range for $k_z > k_{fz}$. Finally, we remark that at wave vectors $k_z < k_{fz}$, the spectrum displays a scaling compatible with a thermal RJ k_z^2 spectrum. In summary, while the prediction $k_h^{-1.69}$ seems to work better than the famous GM spectrum, the agreement is only partial and is limited to some sectors and projections. This fact reflects the complexity of internal

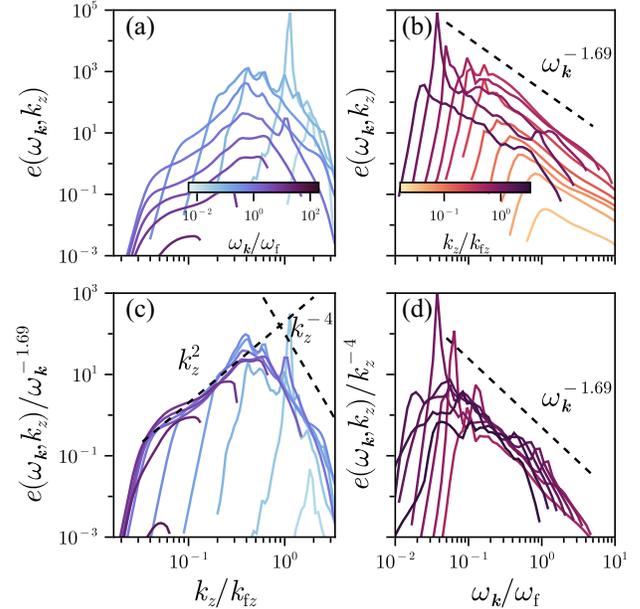


FIG. 5. Slices of the spectrum $e(\omega_k, k_z)$ with fixed ω_k [panels (a) and (c)] and fixed k_z [panels (b) and (d)]. Panels (a), (b) are uncompensated plots, and (c), (d) are compensated by $\omega_k^{-1.69}$ and k_z^{-4} , respectively. In panel (d), we show only lines corresponding to $k_z \gtrsim k_{fz}/2$.

waves and perhaps the lack of pure bihomogeneous power-law steady-state spectrum.

To compare our numerical simulations to oceans and experiments, note that in the WWT framework employed here, $\tau_{nl} = Fr^{-2}\tau_{lin}$, where $\tau_{lin} \sim N^{-1}$ is the characteristic linear time of waves and Fr is the Froude number that compares the typical flow velocity to the restoring velocity of internal waves. Typical values in the ocean and laboratory experiments are $N = 5 \times 10^{-3} \text{ rad.s}^{-1}$ and $N \sim 1 \text{ rad.s}^{-1}$, respectively, whereas the Froude number is about 8.4×10^{-3} for the ocean [52] and 10^{-2} for experiments [53,54]. That leads to a nonlinear time of roughly $\tau_{nl} \sim 33$ days for the ocean and $\tau_{nl} \sim 2.8$ hours for the experiments. Note that the first instabilities transfer energy toward the large scales in about $3\tau_{nl}$ and the condensate peak emerges at about $100\tau_{nl}$. Such time scales are consistent with values found in the literature [29,54]. Yet, it is important to note that our simulation focuses on small-scale forcing, which is different from the large-scale forcing in the ocean.

In this Letter, we have reported numerical simulations of the wave-kinetic equation for weakly nonlinear IGWs. Whereas previous studies have focused on the evaluation of energy transfers for given spectra [14,36,37,41,43,55], we have performed forced-dissipated simulations and looked at the spectrum evolution until it reaches a quasi-steady state.

In our simulations, we have observed the evolution of the energy spectrum under the effect of both local and nonlocal interactions. We showed that for a localized forcing, the nonlocal interactions are initially faster, but the local interactions are important for smoothing out the spectrum on longer timescales. Energy transfers go in every direction of the spectral space: to small scales and small frequencies, but also to large scales and large frequencies. The energy transfer toward the large scales is initially quicker than the transfer toward the small scales. At the final state, a condensate peak emerges in the spectral space at a low value of k_h . The observed dynamics has consequences for the formation of large-scale structures due to interacting waves, and for the small-scale diapycnal mixing. At the end of our simulation, the energy is less present at large frequencies, with a spectrum $e(k_h, k_z) \propto k_h^{-1.69}$ [40,41] in some sectors. However, the k_z slope is different for small k_z and large k_z , and is incompatible with the Garrett-Munk phenomenology and theoretical predictions. It gives another support to the formation of slow modes under the effect of resonant wave-wave interactions. Such an energy transfer toward the large scales could change the energy pathway to mixing and may have to be considered in fine scale parametrization of oceans.

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