



Turbulent pair dispersion as a continuous-time random walk

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The phenomenology of turbulent relative dispersion is revisited. A heuristic scenario is proposed, in which pairs of tracers undergo a succession of independent ballistic separations during time intervals whose lengths fluctuate. This approach suggests that the logarithm of the distance between tracers self-averages and performs a continuous-time random walk. This leads to specific predictions for the probability distribution of separations, which differ from those obtained using scale-dependent eddy-diffusivity models (e.g. in the framework of Richardson's approach). These predictions are tested against high-resolution simulations and shed new light on the explosive separation between tracers.

Key words: mixing, turbulent flows, turbulent mixing

1. Introduction

Tracers in a turbulent flow separate in an explosive manner. Their averaged square distance becomes independent of their initial separation and grows as t^3 at large times. This explains the ability of turbulence to considerably enhance mixing (Dimotakis 2005), but also links to fundamental issues in turbulence, where a key question is to relate the irregularity of the Lagrangian flow with the persistence of a finite dissipation at infinite Reynolds number (Cardy, Falkovich & Gawedzki 2008; Eyink & Drivas 2014). Since the first prediction of the t^3 law by Richardson (1926) and its interpretation in terms of Kolmogorov's similarity hypothesis by Obukhov (1941), precise experimental and numerical measurements and accurate modelling of pair separation have proven to be a particularly laborious exercise, as stressed for instance in the reviews by Sawford (2001) and Salazar & Collins (2009).

A difficulty in observing the explosive law stems from the huge separation of time scales that it requires. Batchelor (1950) indeed showed that the t^3 law is preceded by a ballistic regime during which the mean-square separation is $\propto t^2$. This dominates

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the relative dispersion as long as the initial velocity difference between the tracers has not changed much, that is up to times of the order of the eddy turnover time $\tau_{r_0} \propto r_0^{2/3}$ associated with the inertial-range initial separation r_0 – hereafter referred to as Batchelor’s time scale. The explosive t^3 law takes over at times $t \gg \tau_{r_0}$. While Batchelor’s predictions are quantitatively confirmed in particle-tracking experiments (Berg *et al.* 2006; Ouellette *et al.* 2006) and in direct numerical simulations (Yeung 1994; Sawford, Yeung & Hackl 2008; Bitane, Homann & Bec 2012), the most manifest observations of the t^3 law are limited to initial separations r_0 close to the Kolmogorov dissipative scale η (Ott & Mann 2000; Boffetta & Sokolov 2002; Biferale *et al.* 2005; Eyink 2011). For particles whose initial separation lies in the inertial range, the t^3 growth is more elusive and emerges at best as a short transient for times both much larger than τ_{r_0} and smaller than the integral time scale, from which the separation between tracer particles becomes purely diffusive. Even though the t^3 explosive law can be understood on purely dimensional grounds, there is today a lack of sufficiently accurate data that would substantiate the t^3 law and constrain the underlying mechanisms.

Long-established modelling of relative dispersion is based on the assumption that the distances between tracers undergo a scale-dependent diffusion (Richardson 1926). As stressed by Falkovich, Gawędzki & Vergassola (2001), this presupposes that the system is observed on a time scale t much longer than the Lagrangian correlation time $\tau_r^{\mathcal{L}}$ of the velocity difference between tracers at a distance r . For separations of the order of the typical separation $r \sim \varepsilon^{1/2} t^{3/2}$ (where ε denotes the mean kinetic energy dissipation rate of the turbulent flow), this presumes that $\tau_r^{\mathcal{L}} \ll \varepsilon^{-1/3} r^{2/3} \sim \tau_r$ and thus leads to the unrealistic assumption that the Lagrangian correlation time is much shorter than the Eulerian turnover time. Markovian models involving the joint evolution of separations and velocity differences have been introduced to circumvent such drawbacks (Kurbanmuradov & Sabelfeld 1995; Thomson & Wilson 2013). They are mainly based on the shortness of acceleration correlation times in turbulence and usually rely on the input of Eulerian single-time statistics. Possible pitfalls in their justification relate to the neglect of long-term memory effects due to the persistence of turbulent flows. In an ideal infinite-Reynolds-number turbulent flow, most Markovian models admit scaling solutions of the form $r \propto t^{3/2}$ and $v = (dr/dt) \propto t^{1/2}$. However, at the same time, a small-scale regularization is usually required in order to prevent particle pairs from collapsing together at a finite time with a vanishing velocity difference. The scaling solutions depicted above usually coexist with their ‘dual’ $r \propto (t_* - t)^{3/2}$ with $v \propto -(t_* - t)^{1/2}$. On the one hand, prevention of possible finite-time singularities requires modelling of the dissipative-range physics of turbulence. On the other hand, such events might correspond to the loss of memory, which could justify the applicability of diffusive approaches at sufficiently large times. The recent numerical studies of relative dispersion by Scatamacchia, Biferale & Toschi (2012) and Bitane, Homann & Bec (2013) have clearly stressed the importance of small scales in the overall evolution of distances: some pairs remain trapped at scales $r \ll r_0$ for very long times and give an important contribution to the average separation. The need to account for such trapping events has motivated the introduction of non-Markovian models (Shlesinger, West & Klafter 1987; Faller 1996; Rast & Pinton 2011). Most of them rely on prescribing a distribution of waiting times.

We follow here a slightly different route to account for non-Markovianity. Numerical data from a large-scale simulation of 3-D homogenous turbulence are used, to question the possibility that systematic deviations in the distribution of

the interparticle separations from Richardson’s self-similar solution could stem from a multiplicative process. In §2, we motivate this possibility using a handwaving description of a ballistic phenomenology, observed to be compatible with the bulk of the distribution of distances obtained from numerical data. In §3, we flesh this observation out and describe a piecewise-ballistic phenomenological toy model. This yields a whole family of self-similar distributions for the interparticle distances, and predicts that the distributions of their logarithms should collapse towards a well-defined distribution. Finally, in §4, this prediction is tested against numerical data, and some perspectives are given.

2. The interparticle distance as a multiplicative process?

Most models for relative dispersion reproduce the t^3 explosive law of the long-time behaviour of the mean-square separation. However, they usually yield different probability distributions of the distance r between tracers at time t . Richardson’s eddy-diffusivity approach leads to the self-similar distribution (see Salazar & Collins 2009)

$$p(r, t) = \frac{426}{35} \sqrt{\frac{286}{\pi}} \frac{r^2}{(g\varepsilon)^{3/2} t^{9/2}} \exp \left[-\frac{1}{2} \sqrt[3]{1287} \frac{r^{2/3}}{(g\varepsilon)^{1/3} t} \right], \quad (2.1)$$

which is uniquely determined by Richardson’s constant $g = (r^2)/(\varepsilon t^3)$. The only setting where this distribution is exact is for delta-correlated-in-time flows (Falkovich *et al.* 2001). Be it using massive numerical simulations or sophisticated particle-tracking experiments, one expects to measure some systematic deviations from this distribution. Those deviations exist and have been reported but, quite remarkably, they are rather mild: Richardson’s diffusive mechanism appears to predict the bulk of the distribution correctly and only fails to accurately describe rare events. More precisely, for tracers with an initial distance r_0 of the order of the Kolmogorov scale, Richardson’s distribution seems to overestimate the fraction of particles separating faster than t^3 (Scatamacchia *et al.* 2012). When r_0 lies within the inertial range, it underestimates the fraction of pairs that separate faster than the average (Bitane *et al.* 2013). As those deviations are very fine, one cannot preclude that they are either a consequence of intermittency or of finite-Reynolds-number effects that would contaminate the statistics at both very large and very small scales. A more serious concern exists though: as stated earlier, the eddy-diffusivity framework entails short correlation times for the velocity differences between tracers and thus has debatable physical origins.

In order to remain closer to phenomenological considerations, we propose a completely different mechanism that relies on Batchelor’s ballistic separation of pairs. Let us imagine the following simplified scenario. Two tracers initially separated by a distance r_0 inside the inertial range will follow a ballistic motion with a velocity difference δu_0 for a short period of time τ_0 . By the Kolmogorov 1941 phenomenology (K41) the time and velocity should scale as $\tau_0 \sim r_0^{2/3}$ and $\delta u_0 \sim r_0^{1/3}$. Therefore, after the time τ_0 the pair will be separated by a distance $r_{\tau_0} = r_0 + \tau_0 \delta u_0 = r_0(1 + a_0)$, where a_0 is in principle a scale-independent random variable that accounts for the fluctuations of the flow. Applying the same argument to r_{τ_0} , we find $r_{\tau_0+\tau_1} = r_0(1 + a_0)(1 + a_1)$, where a_1 is independent of a_0 , and so on. Finally, the relative distance between the two tracers at time t is given by $r_t = r_0(1 + a_0) \cdots (1 + a_n)$ with $t = \tau_0 + \cdots + \tau_n$, where by construction the number of terms is $n \sim \ln t$. Although naive, such a piecewise-ballistic phenomenology relies on the physical ideas of Batchelor and

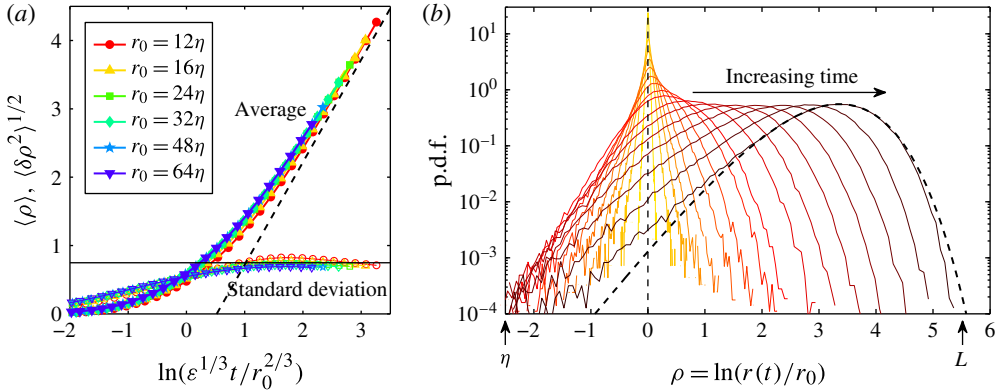


FIGURE 1. (a) Average value and standard deviation of $\rho = \ln(r(t)/r_0)$ for various initial separations r_0 and as a function of $\ln(t/\tau_{r_0})$ with $\tau_{r_0} = \varepsilon^{-1/3} r_0^{2/3}$. The dashed line is $\langle \rho \rangle = (3/2) \ln(t/\tau_{r_0}) + (1/2) \log g - 0.46$ with $g = 0.52$ and the solid line is $\langle \delta \rho^2 \rangle^{1/2} = 0.748$; both correspond to Richardson's self-similar predictions. (b) Distributions of ρ for $r_0 = 12\eta$ and various times from $t \approx 0.05 \tau_{r_0}$ to $12 \tau_{r_0}$ (corresponding to $0.25 \tau_\eta$ and $0.35 \tau_L$, respectively); the two vertical arrows mark the dissipative scale η and the integral scale L . The dashed line is Richardson's distribution.

makes use of the ballistic separation between tracers, whose validity is strongly assessed by numerics and experiments. This scenario suggests that the t^3 law may appear as the consequence of the separations increasing multiplicatively rather than additively. If this was indeed the case, the natural observable would not be the distance but rather its logarithm, which would then be expected to self-average. Indeed, all the n random variables a_i are independent and identically distributed. By the law of large numbers, we thus expect at large times $\ln r_t \propto n \sim \ln t$.

To test those ideas we analyse data from a large direct numerical simulation of 3-D homogeneous turbulence inside a triply periodic box, which uses 4096^3 grid points to achieve a Taylor-based Reynolds number $R_\lambda \simeq 730$ and is seeded with 10^7 tracers (see Bitane *et al.* 2013 for more details). Figure 1(a) shows the evolution of the average of the logarithm $\rho = \ln(r_t/r_0)$ of the interparticle separations, together with its standard deviation. The statistics is conditioned on the initial distances r_0 between the particles, which are chosen to lie within the inertial range (between 12η and 64η). To deal with dimensionless quantities, distances between pairs of tracers are divided by their initial separations, while time scales are divided by the associated Batchelor time scale. At large times the averaged logarithm of the separation exhibits a slant asymptote, whose linear slope cannot be quantitatively distinguished from 3/2: the dashed line shows a behaviour $\langle \rho \rangle \approx (3/2) \ln t + \text{const}$. From the law of large numbers, one also expects that the variance $\langle \delta \rho^2 \rangle = \langle \rho^2 \rangle - \langle \rho \rangle^2$ will increase as $\ln t$. This feature is clearly not observed: the standard deviation of the logarithm rather reaches a plateau for times much longer than the Batchelor time τ_{r_0} . From the latter observation, we cannot exclude that the evolution of the interparticle distances is multiplicative. Yet, such a multiplicative evolution would need to be non-Markovian. In other words, a non-constant time stepping may be involved in the above phenomenology. This would break the assumptions of the law of large numbers.

Figure 1(b) shows the time evolution of the probability distribution of ρ . After a short transient, one observes a self-similar regime, which might explain the behaviours

of both the average and the standard deviation. Indeed, if the distribution of distances takes the scale-invariant form $p(r, t) = r^{-1} \Phi[r/\ell(t)]$, with $\ell(t)$ an arbitrary function of time, then the distribution of $\rho = \ln(r/r_0)$ can be written as $p(\rho, t) = \Psi[\rho - \log(\ell(t)/r_0)]$, with $\Psi[z] = \Phi[e^z]$. This leads to

$$\langle \rho \rangle = \ln(\ell(t)/r_0) + \langle z \rangle, \quad \langle \delta \rho^2 \rangle = \langle \delta z^2 \rangle, \quad (2.2a,b)$$

where the (time-independent) moments of z are obtained by using the distribution $\Psi[z]$. It should be noted that by construction $\langle \log r \rangle$ does not depend on the initial conditioning r_0 . If in addition $\ell(t)$ is a power law of t , then $\langle \rho \rangle \propto \ln t$ and $\langle \delta \rho^2 \rangle \simeq \text{const.}$, as observed in the data. For the Richardson distribution (2.1), $\ell(t) = (g \varepsilon)^{1/2} t^{3/2}$ and, remarkably, the full distribution of z depends neither on the constant g nor on any other physical quantities. Hence, once $\langle \rho \rangle$ is known, the distribution of ρ induced by (2.1) does not require any fitting. The behaviours associated with Richardson's distribution are displayed in figure 1(a) as dashed and solid lines. They give a good approximation for both the mean and the standard deviation of ρ . As seen in figure 1(b), after the transient given by the Batchelor time scale, Richardson's distribution reproduces the bulk well but fails to accurately predict the extreme fluctuations of ρ . This can hardly be blamed on finite-range effects as almost all separations are well inside the inertial range – as indicated by the two vertical arrows.

The most noticeable departure from Richardson's distribution occurs at separations much smaller than their average. We indeed clearly observe $p(\rho) \propto e^{2\rho}$, hereby evidencing a power-law behaviour $p(r) \propto r$ for distances (as already observed by Bitane *et al.* 2013). This is in contrast to the scaling $p(r) \propto r^2$ obtained from eddy-diffusivity models in the K41 framework, including in addition to Richardson's model those where the diffusion coefficient depends also on time (i.e. of the form $D \sim \varepsilon^\alpha r^\beta t^\gamma$, see Henschel & Procaccia 1984). This suggests that the statistics of extreme events cannot be captured by simple diffusive models. However, as we will now see, the heuristic multiplicative approach described earlier can be refined. It leads to a qualitative understanding of the self-similarity and the tails of the distributions observed in figure 1(b).

3. A piecewise-ballistic heuristic scenario

A model that predicts the long-time t^3 explosive law but disregards the short-time ballistic behaviour may yield some biased insights, as it will probably fail to take into account the intrinsic non-Markovian nature of pair separation. Ilyin, Procaccia & Zagorodny (2010) and Eyink & Benveniste (2013) therefore chose to study pair dispersion in terms of a diffusion equation with a memory kernel. We explore here an alternative scenario where we literally implement the ballistic ideas. We argue that this short-time behaviour can be thought of not only as a transient feature but as the central ingredient that yields the explosive regime. To this end, we propose a piecewise-ballistic scenario of tracer separation, described in terms of a toy stochastic model which fits into the general class of so-called continuous-time random walks (CTRWs, see Hughes 1995). The model is rooted in a very intuitive phenomenology, which is in a sense built in to yield the t^3 law. Yet, it also captures non-trivial features of the large-time statistics of the separations between tracers, among which their self-similarity, their explosive nature and a qualitative description of the distributions of extreme events.

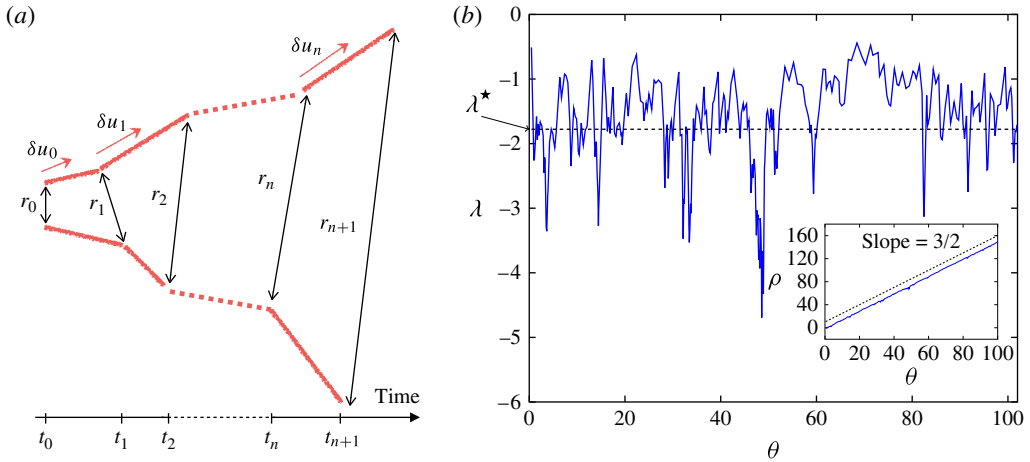


FIGURE 2. (a) Sketch of a piecewise-ballistic scenario. (b) A typical realization of λ as a function of the physical logarithmic time θ , and the corresponding evolution of $\rho = \ln(r/r_0)$ (inset). Here, α is uniform between -1 and 1 , and $\log \beta$ is Gaussian with zero mean and unit variance (please see the text for the definitions).

3.1. Intuitive description of a stochastic piecewise-ballistic model

Given a pair of tracers, we denote by $\mathbf{r}(t)$ and $\delta \mathbf{u}(t)$ their separation and relative velocity, respectively. A ‘ballistic modelling’ consists in assuming that during a time lapse τ , the velocity $\delta \mathbf{u}(t)$ remains constant, so that $\mathbf{r}(t + \tau) \simeq \mathbf{r}(t) + \tau \delta \mathbf{u}(t)$. As demonstrated by Bitane *et al.* (2012) for separations inside the inertial range, the ballistic motion holds typically for a time τ of the order of the time needed to damp out $|\delta \mathbf{u}(t)|^2$ with the average turbulent dissipation rate ε , namely $\tau \simeq |\delta \mathbf{u}(t)|^2 / (2\varepsilon)$. Applying this heuristic argument recursively suggests that the separation between the two tracers undergoes a sequence of non-correlated ballistic increases or decreases at times $t_0 = 0, t_1, t_2, \dots, t_k$ (see figure 2a). Defining the origin of time such that $\mathbf{r}(0) = \mathbf{r}_0$, the separation $\mathbf{r}_k = \mathbf{r}(t_k)$, and the corresponding time t_k will then evolve jointly as

$$\mathbf{r}_{k+1} = \mathbf{r}_k + \frac{|\delta \mathbf{u}_k|^2}{2\varepsilon} \delta \mathbf{u}_k \quad \text{and} \quad t_{k+1} = t_k + \frac{|\delta \mathbf{u}_k|^2}{2\varepsilon}. \quad (3.1a,b)$$

In the language of CTRW, the values (\mathbf{r}_k, t_k) are called the turning points of the process. Between two successive turning points, the separation is by construction ballistic, so that its value at any time can be obtained by linear interpolation. This choice corresponds to a leaping CTRW, as already used by Shlesinger *et al.* (1987) and Sokolov, Klafter & Blumen (2000) in the context of a Levy walk description of turbulent pair dispersion. To entirely describe the statistics of the separations, one now only needs to prescribe statistics (i) for the moduli of the relative velocities $|\delta \mathbf{u}_k|$ and (ii) for the longitudinal components of the velocities $\delta \mathbf{u}_k^{\parallel} = \delta \mathbf{u}_k \cdot \mathbf{r}_k / |\mathbf{r}_k|$. In principle, the distribution of turbulent velocity differences non-trivially depends on the scale $|\mathbf{r}_k|$. This can be due to finite-size effects, so that one has to distinguish between dissipative, inertial and integral scales. We focus here on separations that always remain within the inertial range and exclude any intermittent corrections to the K41 similarity hypothesis. While crude, such an assumption will prove to be

sufficient to reproduce the main mechanisms of relative dispersion. In this framework, the relevant inputs are the statistics of

$$\alpha_k = \frac{\delta \mathbf{u}_k^{\parallel}}{|\delta \mathbf{u}_k|} \quad \text{and} \quad \beta_k = \frac{|\delta \mathbf{u}_k|^3}{2 \varepsilon |\mathbf{r}_k|}. \quad (3.2a,b)$$

To satisfy the 4/5 law, one has to prescribe $\langle \alpha_k^3 \beta_k \rangle = -2/5$, which implies $\langle \alpha_k^3 \rangle < 0$ when the two variables are independent. The variables β_k account for the fluctuations in the rate of energy transfer and, under the K41 assumptions, should be independent of the \mathbf{r}_k . The time lapses between two consecutive turning points may be thought of as correlation times; it is then natural to prescribe that the (α_k, β_k) corresponding to different k are independently distributed. We later refer to the distributions of the noises α_k and β_k as α and β – without a subscript – and denote by $\langle \cdot \rangle$ the average over their realizations.

Under these assumptions, the dynamics of the distance $r_k = |\mathbf{r}_k|$ reduces to

$$r_{k+1} = r_k (1 + 2 \alpha_k \beta_k + \beta_k^2)^{1/2} \quad \text{and} \quad t_{k+1} = t_k + (2 \varepsilon)^{-1/3} (\beta_k r_k)^{2/3}. \quad (3.3a,b)$$

Within this modelling, for a small enough value of t , one observes by construction Batchelor’s t^2 growth for the interparticle separations. This is not the case for finite values of t , due to the multiplicative aspects of the model. However, it should be noted that this process is not purely multiplicative. As discussed in the previous section, if the time increments $\tau_k = t_{k+1} - t_k$ were constant, then the distributions of the logarithms of the distance at a given observation time would evolve towards a Gaussian distribution given by the central limit theorem. This is, however, not the case here, as the time associated with a given pair of tracers is itself a random variable, which is neither additive nor multiplicative. In the present paper, we do not need to prescribe further the distributions of α and β . The only constraint concerns the quantity $\langle \ln(1 + 2\alpha\beta + \beta^2) \rangle$. It is required to be positive in order to ensure that the times t_k go to infinity as the number of turning points diverges. This prevents the sequence t_k from converging and the two tracers from touching each other in a finite time.

3.2. Statistics of the separations from the piecewise-ballistic perspective

We shall not here attempt to work out in full mathematical detail the statistics of the separations that the model predicts. Rather, we focus on a general and qualitative description of these, based on simple physical arguments.

3.2.1. Self-similarity

The piecewise-ballistic scenario as modelled by the system (3.3) yields a whole family of self-similar distributions for the separations. To understand the origin of this self-similarity, it is convenient to introduce the non-dimensional ‘logarithmic’ variables

$$\theta_k = \ln(t_k/\tau_0), \quad \rho_k = \ln(r_k/r_0) \quad \text{and} \quad \lambda_k = \ln(r_k/r_0) - (3/2) \ln(t_k/\tau_0), \quad (3.4a-c)$$

where $\tau_0 = (2 \varepsilon)^{-1/3} r_0^{2/3}$ is a characteristic time lapse associated with the initial separation r_0 . The largest values of λ indicate pairs that separate faster than the typical explosive separation, while large negative values are obtained when they get close to each other. We next define $p_\theta(\rho)$ as the probability density of ρ conditioned

on θ , regardless of the number of turning points needed to reach this time, and write $\langle \cdot \rangle_\theta$ for the corresponding average.

Self-similarity is achieved if the statistics of the variable λ_k becomes steady at long times, e.g. when $\theta \gg 1$. A rigorous proof that λ_k evolves towards a stationary distribution p_∞ goes beyond the scope of the present paper. Still, let us give qualitative arguments, starting from the evolution of θ_k and λ_k . The combination of (3.1) and (3.4) yields

$$\lambda_{k+1} = \lambda_k + \frac{3}{2} \ln \frac{(1 + 2\alpha_k \beta_k + \beta_k^2)^{1/3}}{1 + \beta_k^{2/3} e^{(2/3)\lambda_k}} \quad \text{and} \quad \theta_{k+1} = \theta_k + \ln(1 + \beta_k^{2/3} e^{(2/3)\lambda_k}). \quad (3.5a,b)$$

The equation for λ_k is closed as it does not involve the time variable θ_k . A typical realization of the λ_k values is shown in figure 2(b). They fluctuate around the specific value λ^* , univocally defined by

$$\langle \ln(1 + 2\alpha\beta + \beta^2) \rangle = 3 \langle \ln(1 + \beta^{2/3} e^{2\lambda^*/3}) \rangle. \quad (3.6)$$

It is easily shown that λ^* is always negative. The evolution of λ_k can then be decomposed as the sum of a noise \mathcal{W} with zero mean and a restoring stochastic force \mathcal{F} , namely

$$\lambda_{k+1} = \lambda_k + \mathcal{W}(\alpha_k, \beta_k) + \mathcal{F}(\lambda_k, \beta_k), \quad \text{with} \quad \begin{cases} \mathcal{W}(\alpha, \beta) = \frac{3}{2} \ln \frac{|1 + 2\alpha\beta + \beta^2|^{1/3}}{1 + \beta^{2/3} e^{2\lambda^*/3}}, \\ \mathcal{F}(\lambda, \beta) = \frac{3}{2} \ln \frac{1 + \beta^{2/3} e^{2\lambda^*/3}}{1 + \beta^{2/3} e^{2\lambda/3}}. \end{cases} \quad (3.7)$$

The force $\mathcal{F}(\lambda, \beta)$ is a decreasing function of λ and changes sign at $\lambda = \lambda^*$. We therefore expect λ^* to be a recurrent point. This is the source of the stationarity of the process λ_k , which we have observed numerically. The stationarity of λ_k has several consequences. First, the average logarithmic separation grows asymptotically when $\theta \gg 1$ as $\langle \rho \rangle_\theta = 3\theta/2 + \langle \lambda \rangle_\infty + o(1)$ (see inset of figure 2b), while its variance becomes constant, as observed earlier in our data. Second, the stationarity of λ_k implies an explosive behaviour of separations. The initial separation r_0 does not appear in the dynamics (3.5). Also, one can easily see that the specific choice of τ_0 ensures that r_0 can be simplified in the definition of λ_k . The dependence upon the initial separation is thus entailed in the definition of θ_k . However, this dependence disappears when $t \rightarrow \infty$, so that the stationary distribution of λ_k is independent of r_0 . Finally, for sufficiently large times the probability density of the logarithmic separations is simply translated by the dynamics around its average value: $p_\theta(\rho) = p_\infty(\rho - 3\theta/2)$. This implies that, at large times, the statistics of the distance r approaches the self-similar form $p(r, t) \simeq r^{-1} p_\infty[\ln(r/(\varepsilon^{1/2} t^{3/2}))]$, which is independent of r_0 .

3.2.2. Tails

The piecewise-ballistic mechanism does not yield a single distribution but a whole family of self-similar distributions p_∞ , depending on the choices of α and β . We can, however, try to characterize the tails of p_∞ .

The right-end tail ($\lambda > \lambda^*$) corresponds to tracers that separate faster than the average. At large times, the piecewise model (3.5) predicts that $\lambda(\theta)$ is negative or, in other words, that the logarithmic distance ρ is strictly smaller than three halves

of the logarithmic time. It is indeed easily checked that if λ is negative at a given turning point, then it remains negative afterwards. As λ ends up fluctuating around $\lambda^* < 0$, it is almost certainly negative at large times. A noticeable consequence is that the limiting self-similar distributions have a right-end cutoff. Such a behaviour is in contrast to that obtained from eddy-diffusivity models (including Richardson's), whose tails fall as a double exponential at large values.

The other tail ($\lambda \rightarrow -\infty$) captures the statistics of pairs that are not separated. Numerical simulations of the model reveal that its form does strongly depend on the distribution of α and β . To understand this behaviour further, let us define $\Gamma = \langle \mathcal{F}(-\infty, \beta) \rangle$ and $\kappa = \langle \mathcal{W}(\alpha, \beta)^2 \rangle^{1/2}$, and describe two asymptotic regimes $\kappa \ll \Gamma$ and $\kappa \gg \Gamma$.

- (i) The case $\kappa \ll \Gamma$ describes a situation where tracers almost never come close to each other. When they do, they are immediately pulled back to their initial separation by the restoring force \mathcal{F} . In this case, one-step excursions dominate the left-end tail of the statistics, which is therefore entirely determined by the distribution of \mathcal{W} .
- (ii) The case $\Gamma \ll \kappa$ is opposite. Here, two particles need to undergo a large number of ballistic steps to be pulled back towards λ^* . The time of each step is on average smaller than the typical correlation time $\Delta\theta^* = \langle \log(1 + \beta^{2/3} e^{2\lambda^*/3}) \rangle$ at $\lambda = \lambda^*$. In this limit, the noise is dominant and the discrete dynamics can be approximated by a Brownian motion with a positive drift. One can check that this yields a stationary distribution whose left-end tail is $\propto e^{(2/3+\gamma)\lambda}$, where $\gamma > 0$ depends on the noise kernels. With an accurate choice of the noises, the left exponential slope is therefore likely to be shallower than Richardson's left exponential slope of 3, as suggested by the numerical data.

4. Scaling of the distribution of distances

At this point, one may wonder whether the piecewise-ballistic phenomenology achieves a better description of the full statistics than Richardson's distribution. As the precise shape of the statistics obtained from the piecewise-ballistic scenario is noise-dependent, a detailed answer would require us to plug into our stochastic model some 'realistic noises' for the distributions of α and β . This goes beyond our intention, since we consider here the set of (3.1) as a 'scenario' rather than a genuine 'modelling' of the separations.

To our taste, the main virtue of the model is that it provides a non-Markovian physical interpretation of the explosive nature of the interparticle separations, namely the independence from the initial separation. We believe that dynamical memory effects are essential for the understanding of extreme events in relative dispersion. One may therefore wonder whether such effects are universal and whether their signature is intrinsically linked to the explosive separation. A related question concerns the origin of the observed deviations from Richardson's self-similar distribution. Are they due to finite-size effects, to the intermittency of velocity statistics or rather, as we think, to the limits of the eddy-diffusivity approach? Our impression is that this question has been somehow overlooked in the previous literature. Experimental and numerical datasets are often confronted with Richardson's distribution using a linlog representation of the distance neighbour function $p(r, t)/(4\pi r^2)$ in such a way that Richardson's distribution appears as a straight line (see, e.g. Ouellette *et al.* 2006). Such a representation puts a visual emphasis on the collapse of the bulk of the distribution, but is not optimal for a thorough study of the tails. In figure 3, we use

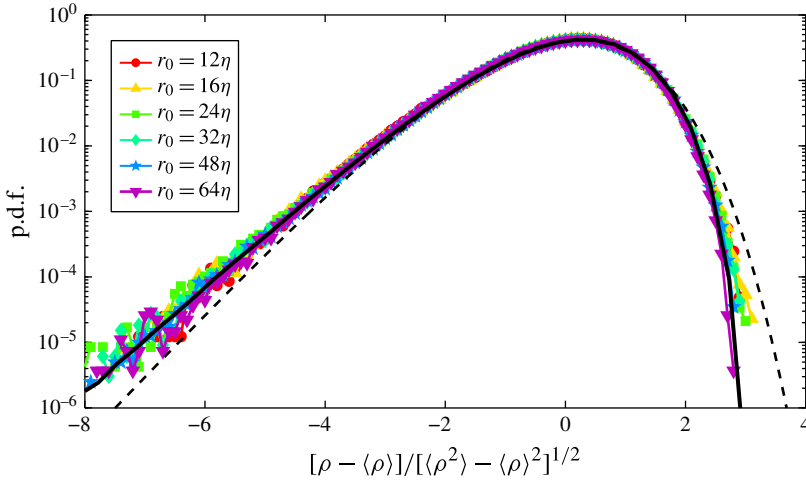


FIGURE 3. The probability density function of the logarithm of the interparticle distance $\rho = \ln(r(t)/r_0)$ for various r_0 and, in each case, at time $t \approx 9 \tau_{r_0}$ with $\tau_{r_0} = \varepsilon^{-1/3} r_0^{2/3}$. The dashed line is Richardson's distribution. The black solid line is the steady distribution obtained numerically from the ballistic model for the arbitrary choices $p(\alpha) = (5/6)((\alpha + 1)/2)^{2/3}$, $p(\beta) = \exp[-(\ln \beta)^2/2]/(\sqrt{2\pi}\beta)$, and with the two variables independent from each other.

linlog coordinates to represent the distributions of the rescaled logarithmic distances $\tilde{\rho} = [\rho - \langle \rho \rangle] / [(\rho^2) - \langle \rho \rangle^2]^{1/2}$ observed in the numerical simulation for six different initial separations r_0 inside the inertial range and for each at the fixed time $t \approx 9 \tau_{r_0}$. With such a choice, the distribution of the logarithmic distances seems to converge towards a single distribution, regardless of r_0 . The collapse of the full distribution makes the explosive nature of pair separation explicit. It is also once again clear that the limiting distribution is not Richardson's (dashed line). It should be noted that a casual choice for the statistics of α and β makes the piecewise-ballistic steady distribution (solid line) fit the data better. It could be interesting to use experimental data to discriminate some realistic distributions. The model predicts a sharp cutoff at large values. However, the current statistical accuracy does not enable us to discriminate between such a behaviour and the double exponential obtained in the framework of eddy-diffusivity approaches.

To conclude, let us stress again that the piecewise-ballistic phenomenology provides a new and intuitive way of thinking about the problem of pair dispersion and reproduces some salient statistical features of tracer separation. While it could also be used to investigate possible effects of the fluid flow intermittency, we limited our study here to the K41 framework. Another interesting development would be to use such a model to shed some light on the asymmetry between forward and backward in time turbulent dispersion (Benveniste & Drivas 2014). This symmetry breaking is expected to relate to non-trivial correlations between α and β . The proposed toy model displays a number of general trends that include (i) the explosive nature of the statistics or, in other words, the property that the steady distributions do not depend on the initial separation; (ii) their self-similarity, which makes the statistics of the logarithm of the separation collapse towards a single distribution; (iii) the presence of a right-end cutoff in the associated probability density; (iv) the growth of the average

of the logarithmic separation as three halves of the logarithmic time, compatible with the t^3 law, resulting from the multiplicative nature of the separation process.

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References

- BATCHELOR, G. K. 1950 The application of the similarity theory of turbulence to atmospheric diffusion. *Q. J. R. Meteorol. Soc.* **76**, 133–146.
- BENVENISTE, D. & DRIVAS, T. D. 2014 Asymptotic results for backwards two-particle dispersion in a turbulent flow. *Phys. Rev. E* **89**, 041003.
- BERG, J., LÜTHI, B., MANN, J. & OTT, S. 2006 Backwards and forwards relative dispersion in turbulent flow: an experimental investigation. *Phys. Rev. E* **74**, 016304.
- BIFERALE, L., BOFFETTA, G., CELANI, A., DEVENISH, B. J., LANOTTE, A. & TOSCHI, F. 2005 Lagrangian statistics of particle pairs in homogeneous isotropic turbulence. *Phys. Fluids* **17**, 115101.
- BITANE, R., HOMANN, H. & BEC, J. 2012 Timescales of turbulent relative dispersion. *Phys. Rev. E* **86**, 045302.
- BITANE, R., HOMANN, H. & BEC, J. 2013 Geometry and violent events in turbulent pair dispersion. *J. Turbul.* **14** (2), 23–45.
- BOFFETTA, G. & SOKOLOV, I. M. 2002 Relative dispersion in fully developed turbulence: the Richardson's law and intermittency corrections. *Phys. Rev. Lett.* **88**, 094501.
- CARDY, J., FALKOVICH, G. & GAWĘDZKI, K. 2008 *Non-Equilibrium Statistical Mechanics and Turbulence*, London Mathematical Society Lecture Note Series, vol. 355. Cambridge University Press.
- DIMOTAKIS, P. E. 2005 Turbulent mixing. *Annu. Rev. Fluid Mech.* **37**, 329–356.
- EYINK, G. L. 2011 Stochastic flux freezing and magnetic dynamo. *Phys. Rev. E* **83**, 056405.
- EYINK, G. L. & BENVENISTE, D. 2013 Diffusion approximation in turbulent two-particle dispersion. *Phys. Rev. E* **88** (4), 041001.
- EYINK, G. L. & DRIVAS, T. D. 2014 Spontaneous stochasticity and anomalous dissipation for Burgers equation. Preprint, [arXiv:1401.5541](https://arxiv.org/abs/1401.5541).
- FALKOVICH, G., GAWĘDZKI, K. & VERGASSOLA, M. 2001 Particles and fields in fluid turbulence. *Rev. Mod. Phys.* **73**, 913–975.
- FALLER, A. J. 1996 A random-flight evaluation of the constants of relative dispersion in idealized turbulence. *J. Fluid Mech.* **316**, 139–161.
- HENTSCHEL, H. G. E. & PROCACCIA, I. 1984 Relative diffusion in turbulent media: the fractal dimension of clouds. *Phys. Rev. A* **29**, 1461–1470.
- HUGHES, B. D. 1995 *Random Walks and Random Environments*. vol. 1, Clarendon.
- ILYIN, V., PROCACCIA, I. & ZAGORODNY, A. 2010 Stochastic processes crossing from ballistic to fractional diffusion with memory: exact results. *Phys. Rev. E* **81**, 030105.
- KURBANMURADOV, O. & SABELFELD, K. 1995 Stochastic Lagrangian models of relative dispersion of a pair of fluid particles in turbulent flows. *Monte Carlo Meth. Applic.* **1**, 101–136.
- OBUKHOV, A. M. 1941 On the distribution of energy in the spectrum of turbulent flow. *Izv. Akad. Nauk SSSR, Ser. Geogr. Geofiz.* **5**, 453–466.
- OTT, S. & MANN, J. 2000 An experimental investigation of the relative diffusion of particle pairs in three-dimensional turbulent flow. *J. Fluid Mech.* **422**, 207–223.
- OUELLETTE, N. T., XU, H., BOURGOIN, M. & BODENSCHATZ, E. 2006 An experimental study of turbulent relative dispersion models. *New J. Phys.* **8**, 109 (1–23).

- RAST, M. P. & PINTON, J.-F. 2011 Pair dispersion in turbulence: the subdominant role of scaling. *Phys. Rev. Lett.* **107**, 214501.
- RICHARDSON, L. F. 1926 Atmospheric diffusion shown on a distance–neighbour graph. *Proc. R. Soc. Lond. A* **110**, 709–737.
- SALAZAR, J. P. L. C. & COLLINS, L. R. 2009 Two-particle dispersion in isotropic turbulent flows. *Annu. Rev. Fluid Mech.* **41**, 405–432.
- SAWFORD, B. L. 2001 Turbulent relative dispersion. *Annu. Rev. Fluid Mech.* **33**, 289–317.
- SAWFORD, B. L., YEUNG, P. K. & HACKL, J. F. 2008 Reynolds number dependence of relative dispersion statistics in isotropic turbulence. *Phys. Fluids* **20** (6), 065111.
- SCATAMACCHIA, R., BIFERALE, L. & TOSCHI, F. 2012 Extreme events in the dispersions of two neighboring particles under the influence of fluid turbulence. *Phys. Rev. Lett.* **109**, 144501.
- SHLESINGER, M. F., WEST, B. J. & KLAFTER, J. 1987 Lévy dynamics of enhanced diffusion: application to turbulence. *Phys. Rev. Lett.* **58** (11), 1100–1103.
- SOKOLOV, I. M., KLAFTER, J. & BLUMEN, A. 2000 Ballistic versus diffusive pair dispersion in the Richardson regime. *Phys. Rev. E* **61** (3), 2717–2722.
- THOMSON, D. J. & WILSON, J. D. 2013 History of Lagrangian stochastic models for turbulent dispersion. In *Lagrangian Modeling of the Atmosphere*, pp. 19–36. Wiley Online Library.
- YEUNG, P. K. 1994 Direct numerical simulation of two-particle relative diffusion in isotropic turbulence. *Phys. Fluids* **6** (10), 3416–3428.