

LETTER • **OPEN ACCESS**

Strong and weak wave turbulence regimes in Bose–Einstein condensates

To cite this article: Ying Zhu *et al* 2026 *Rep. Prog. Phys.* **89** 030501

View the [article online](#) for updates and enhancements.

You may also like

- [Steady-states and response functions of the periodically driven \$O\(N\)\$ scalar field theory](#)
Oriana K Diessel, Subir Sachdev and Pietro M Bonetti
- [How semiconducting are ferroelectrics: the fundamental, optical and transport gaps of \$\text{Na}_{1-x}\text{Bi}_x\text{TiO}_3\$ – \$\text{BaTiO}_3\$ and \$\text{NaNbO}_3\$](#)
Pengcheng Hu, Nicole Bein, Chinmay Chandan Parhi et al.
- [Probing flying-focus wakefields](#)
Aaron Liberman, Anton Golovanov, Sheroy Tata et al.

Reports on Progress in Physics

**LETTER****OPEN ACCESS****RECEIVED**
4 December 2025**REVISED**
17 February 2026**ACCEPTED FOR PUBLICATION**
6 March 2026**PUBLISHED**
11 March 2026

Original content from this work may be used under the terms of the [Creative Commons Attribution 4.0 licence](#).

Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.



Strong and weak wave turbulence regimes in Bose–Einstein condensates

Ying Zhu^{1,*} , Giorgio Krstulovic^{1,2,*} and Sergey Nazarenko¹ ¹ Université Côte d'Azur, CNRS, Institut de Physique de Nice (INPHYNI), 17 rue Julien Lauprêtre, 06200 Nice, France² Université Côte d'Azur, Observatoire de la Côte d'Azur, CNRS, Laboratoire Lagrange, Boulevard de l'Observatoire CS 34229, F 06304 Nice Cedex 4, France

* Authors to whom any correspondence should be addressed.

E-mail: zhuying0325@gmail.com and krstulovic@oca.eu**Keywords:** Bose–Einstein condensates, wave turbulence, quantum turbulence, strong turbulence

Corresponding editor: Dr Paul Mabey

Abstract

When a turbulent Bose–Einstein condensate (BEC) is driven out-of-equilibrium at a scale much smaller than the system size, nonlinear wave interactions transfer particles towards large scales in an inverse cascade process. In this work, we numerically study wave turbulence in a three-dimensional BEC in forced and dissipative inverse cascade settings. We observe that when the forcing rate increases, thereby increasing the particle flux, the turbulence spectrum gradually transitions from the weak-wave Kolmogorov–Zakharov cascade to a critical balance state characterized by a range of scales with balanced linear and nonlinear dynamic timescales. Further forcing increases lead to a coherent condensate component superimposed with Bogoliubov-type acoustic turbulence. The role of vortices in such a strongly forced state is marginal, which makes this new state distinct from the strongly turbulent state composed of a tangle of quantized vortex lines. We then use our predictions and numerical data to formulate a new out-of-equilibrium equation of state for the 3D inverse cascade.

1. Introduction

Turbulence is a central example of strongly non-equilibrium systems whose state is determined by an energy (or another invariant) flux through scales. When forcing and dissipation scales are well separated, an inertial range of scales emerges, where the dynamics is driven by the nonlinear interactions only. The central concept of turbulence is the idea of cascade, originally proposed by Richardson in the case of hydrodynamic turbulence. In 1941, Kolmogorov pushed this concept further by postulating that in the limit of an infinite inertial range, the energy dissipation rate ε is finite, necessarily equal to the energy flux, and it is the only dimensional parameter involved in the turbulent energy cascade. A simple dimensional analysis then suffices to fix scaling exponents, which leads to the famous Kolmogorov energy spectrum $E_k \propto \varepsilon^{2/3} k^{-5/3}$, with k the wavevector [1]. Many other turbulent systems exist where more than one dimensional parameter comes into play, so scaling exponents cannot be fixed so simply. Most typical examples are magnetohydrodynamics and nonlinear wave systems such as internal and inertial waves in the oceans and atmospheres, capillary-gravity waves on the surface of liquids, or Bose–Einstein condensates (BECs) [2–5]. Fortunately, when nonlinearity is small, the weak wave turbulence theory (WWTT) predicts the whole spectrum, including universal dimensionless pre-factors [6]. When nonlinearity becomes strong, only phenomenological arguments can be provided in rare cases. How a system transitions from the weak to the strong wave turbulence (WT) regimes remains an open question.

In this Letter we study the weak-to-strong transition in turbulence of BECs. BECs provide an excellent experimental platform for studying turbulence in general and WT in particular because of the versatility of optical techniques in creating experimental setups and diagnostics. Several types of BEC turbulence setups in two and three dimensions have been realized by various types of forcing, from shaking

the BEC trap to dragging repulsive potentials through the system, thereby generating quantized vortices and/or dispersive waves [7–16]. Recently, experimentalists using a shaking trap have succeeded in creating turbulent BECs incorporating a synthetic small-scale dissipation mechanism [11, 13, 14]. Introducing such a synthetic dissipation is an important new element as it allows for producing stationary BEC turbulence and, therefore, testing theoretical predictions for the statistically steady state. In particular, a stationary WT state corresponding to a direct local cascade of energy from low momenta (where it was forced) to high momenta (where it was dissipated) was reported in [11]. The resulting spectrum was explained theoretically and confirmed numerically in [17]. The latter paper has also derived and simulated the stationary inverse WT cascade directed from a source at high momenta to a dissipation range at low momenta—a setup yet to be implemented experimentally. The present work extends this study for a very wide range of forcing amplitudes, including cases far beyond the WWTT applicability.

To model BECs, we use the Gross–Pitaevskii equation (GPE) for the complex wave function $\psi(\mathbf{x}, t)$,

$$i\frac{\partial\psi}{\partial t} = -\alpha\nabla^2\psi + \beta|\psi|^2\psi, \quad (1)$$

where $\alpha = \hbar/2m$, $\beta = g/\hbar$ and $g = 4\pi\hbar^2 a_s/m$, with \hbar the reduced Planck constant, m the boson mass, and a_s the s -wave scattering length [18]. For simplicity, we consider a triply periodic cube of side L and volume $V = L^3$. The GPE (1) conserves the total number of particles and energy per unit volume,

$$N = \frac{1}{L^3} \int |\psi(\mathbf{x}, t)|^2 d\mathbf{x}, \quad (2)$$

$$H = \frac{\hbar}{L^3} \int \left[\alpha |\nabla\psi(\mathbf{x}, t)|^2 + \frac{\beta}{2} |\psi(\mathbf{x}, t)|^4 \right] d\mathbf{x}, \quad (3)$$

respectively. In a turbulent setting, energy is transferred towards small scales (direct cascade), whereas particles exhibit an inverse cascade towards large scales. Most of the turbulent BEC experiments have studied the direct energy cascade [11, 13], whereas inverse particle turbulent transfers have been recently observed both in 2D and 3D experiments [14, 19]. (The inverse particle cascade should not be confused with the inverse energy cascades and clustering in the 2D BEC quantized vortex turbulence studied in [15, 16]).

In this Letter, we focus on the inverse particle cascade in 3D. In particular, we study how the system transitions from weak to strong WT as the particle flux per unit volume $|Q_0|$ is increased. The advantage of this setting is that the WWTT provides an exact and simple analytical prediction, unlike the 3D direct energy cascade or the cascades in 2D BECs that present certain mathematical challenges.

2. Weak WT and critical balance predictions

As usual in turbulence, let us start with dimensional considerations. The main observable in the turbulent cascade is the waveaction spectrum, $n_{\mathbf{k}}(t) \equiv n(\mathbf{k}, t) = \frac{V}{(2\pi)^3} \langle |\hat{\psi}_{\mathbf{k}}(t)|^2 \rangle$, where $\hat{\psi}_{\mathbf{k}} = \frac{1}{L^3} \int \psi(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}$ is the Fourier transform of $\psi(\mathbf{x})$, and the brackets denote the average over fluctuations. Note that $n_{\mathbf{k}}$ is dimensionless, and from equation (1), $[\alpha] = \text{length}^2/\text{time}$ and $[\beta] = \text{length}^3/\text{time}$. As $[Q_0] = \text{length}^{-3}/\text{time}$, then by matching dimensions we have

$$n_{\mathbf{k}} = C \left(\frac{|Q_0|\beta^5}{\alpha^6} \right)^y \left(\frac{\alpha}{\beta k} \right)^x, \quad (4)$$

where C is a dimensionless constant, the two terms in parentheses are independent dimensionless combinations, and we assumed a power-law dependence on these parameters with x and y being *a priori* arbitrary exponents. Here, x fixes the dependence of the waveaction spectrum on scales, whereas y determines the behavior of the so-called out-of-equilibrium equation of state (EoS) [20, 21]. Indeed, the spectrum's amplitude dependence on the flux Q_0 is analogous to the dependence of gas density on the temperature at thermal equilibrium (e.g. at constant pressure). The WWTT predicts the Kolmogorov–Zakharov (KZ) spectrum that fixes all the exponents and the universal dimensionless constant C [17]:

$$n_{\mathbf{k}} = C_i \frac{\alpha^{1/3}}{\beta^{2/3}} |Q_0|^{1/3} k^{-7/3}, \quad C_i \approx 7.5774045 \times 10^{-2}. \quad (5)$$

This spectrum is an exact solution of the four-wave kinetic equation associated with the GPE, which is the central equation of the WWTT obtained under the assumptions of small amplitudes and random

phases of waves. Accordingly, the constant C_i is given by an analytical expression, the approximate value of which is provided above. It is a WT analog of the Kolmogorov constant in the famous Kolmogorov spectrum of classical hydrodynamic turbulence [1]. The KZ spectrum (5) was found to be in excellent agreement with numerical simulations [17].

When the nonlinearity is strong, phases might be strongly correlated, and the spectrum could be guessed by invoking the critical balance (CB) assumption where linear and nonlinear timescales are balanced at each scale within a certain range of scales, a concept introduced in [22] in the context of the magnetohydrodynamic turbulence. In fact, approaches similar in spirit were independently proposed in various physical contexts (see examples in [23]) with the first such theory developed in the context of water waves, which led to the famous Phillips spectrum [24]. The appearance of CB can often be understood as an effect of a set of coherent structures with a range of sizes generated in the system (possibly tightly packed in physical space). For example, the Phillips spectrum is often attributed to sharp-crested Stokes waves, which are limited to a critical amplitude by wave breaking [25]. It is natural to think that in the GPE system, such CB states could be characterized by a set of densely packed solitonic structures of various sizes. For GPE, the CB argument was first introduced in [26], and here we will revise and refine it. We start with the GPE (1) written in Fourier space

$$i\frac{\partial\hat{\psi}_{\mathbf{k}}}{\partial t} = \omega_k\hat{\psi}_{\mathbf{k}} + \beta\sum_{123}\hat{\psi}_1^*\hat{\psi}_2\hat{\psi}_3\delta_{23}^{k_1}, \text{ with } \omega_k = \alpha k^2, k = |\mathbf{k}|, \quad (6)$$

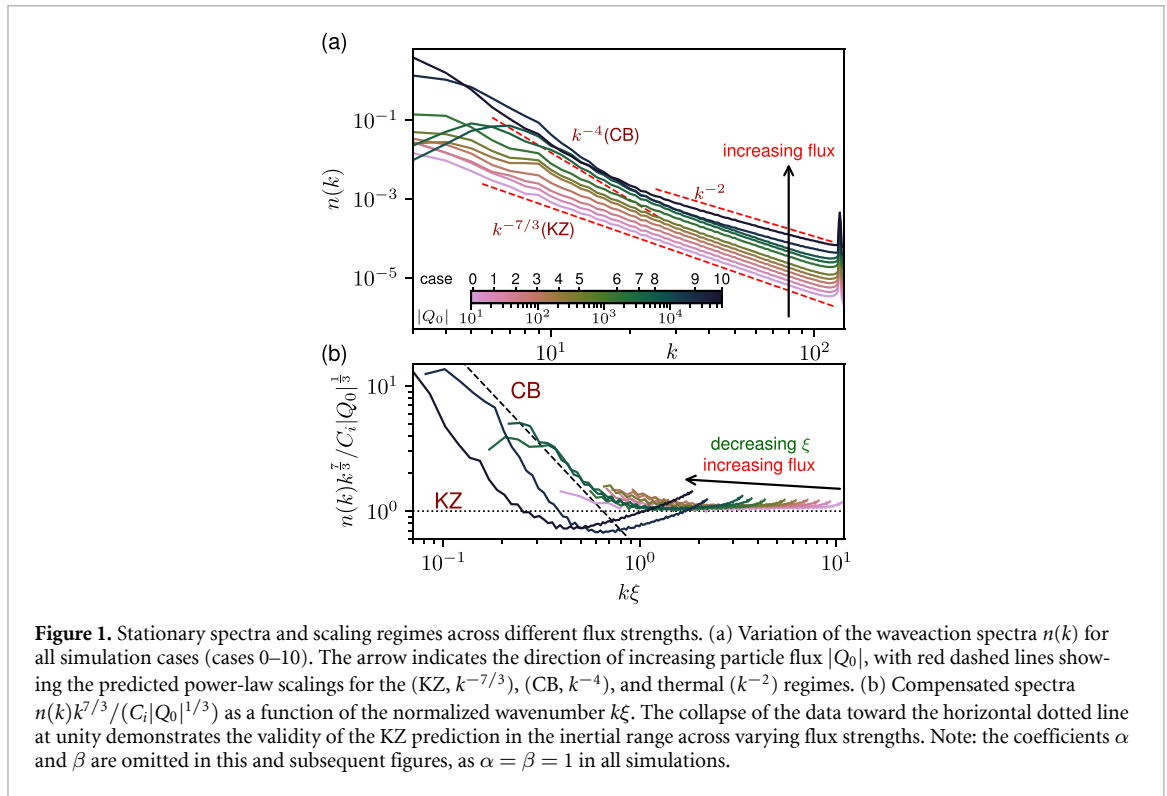
where \sum_{123} stands for a sum over all wave vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$, $\psi_i = \psi(\mathbf{k}_i)$, and $\delta_{23}^{k_1}$ is a Kronecker delta: 1 if $\mathbf{k} + \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$, 0 otherwise. Note that for both WWTT and CB, the leading-order nonlinear effect in (6) is provided by the diagonal terms in the sum, i.e. those with $\mathbf{k} = \mathbf{k}_2, \mathbf{k}_1 = \mathbf{k}_3$ and $\mathbf{k} = \mathbf{k}_3, \mathbf{k}_1 = \mathbf{k}_2$. They lead to a nonlinear frequency shift $\omega_k \rightarrow \omega_k + 2\beta N$, but do not affect the evolution of $|\hat{\psi}_{\mathbf{k}}|$ and, therefore, do not lead to any spectral redistribution. They can be eliminated by passing into the rotating frame, $\hat{\psi}_{\mathbf{k}} \rightarrow e^{-2i\beta N t}\hat{\psi}_{\mathbf{k}}$. Applying the CB argument to (6), we find $\omega_k|\hat{\psi}_{\mathbf{k}}| \sim \beta|\hat{\psi}_{\mathbf{k}}|^3(k/k_{\text{cor}})^6$ where k_{cor} is the correlation distance in k -space (it is vanishingly small in the WWTT). The $(k/k_{\text{cor}})^6$ -term estimates the number of modes interacting with mode k . With $n_{\mathbf{k}} = (L/2\pi)^3\langle|\hat{\psi}_{\mathbf{k}}|^2\rangle$, this leads to $n_{\mathbf{k}} \sim \frac{\alpha}{\beta}\frac{k_{\text{cor}}^6}{k^4}\frac{L^3}{(2\pi)^3}$. In the limit $L \rightarrow \infty$, this expression should be independent of L . If we additionally postulate that k_{cor} is independent of k then the most general expression for it (leading to an L -independent spectrum) is $k_{\text{cor}} \sim \left(\frac{|Q_0|\beta^5}{\alpha^6}\right)^{y/6}\left(\frac{2\pi\alpha}{L\beta}\right)^{1/2}$. This finally gives

$$n_{\mathbf{k}} = C\left(\frac{|Q_0|\beta^5}{\alpha^6}\right)^y\left(\frac{\alpha}{\beta k}\right)^4 \quad (7)$$

i.e. equation (4) with $x=4$. Note that this value is a consequence of our assumption that k_{cor} is k -independent. Unfortunately, the CB argument is too basic to be able to predict the exponent y , i.e. to determine how k_{cor} scales with the flux. We will test the scaling $x=4$ and extract the value of y using the numerical simulations.

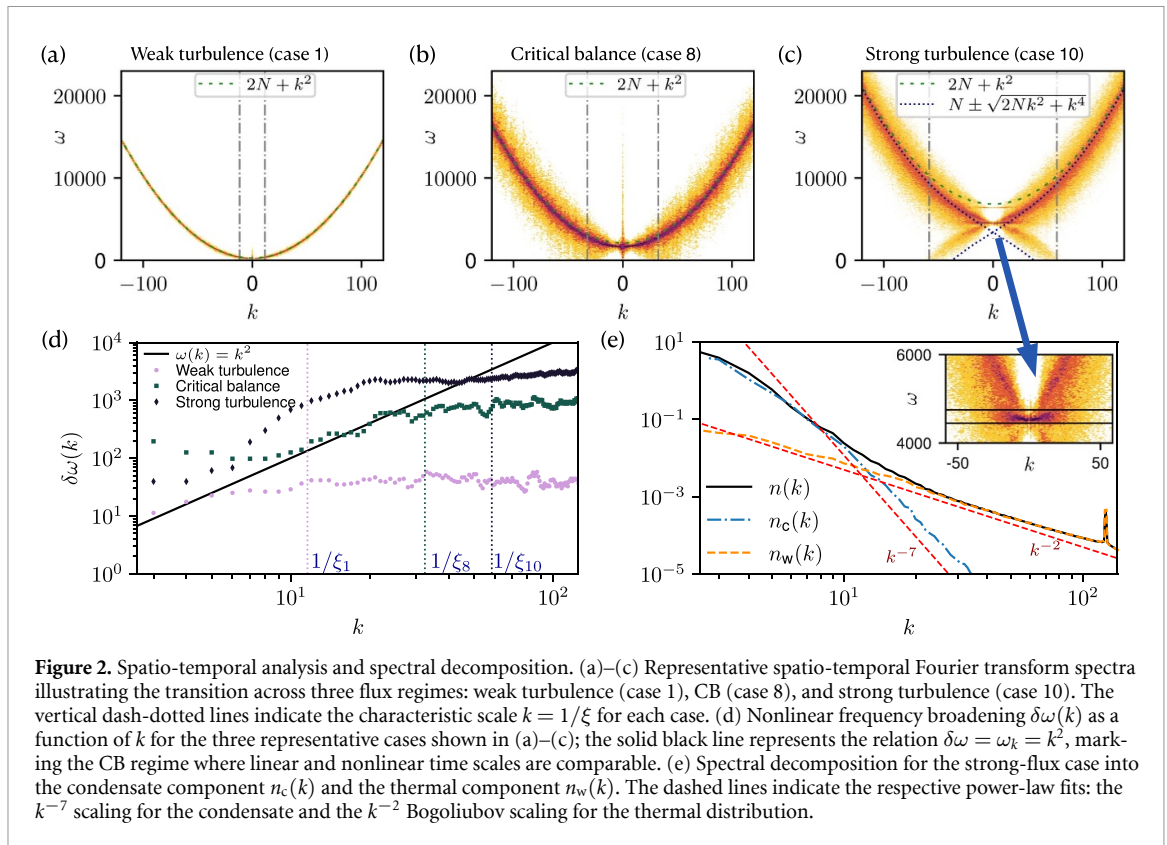
3. Numerical simulations

We perform numerical simulations of the forced-dissipated GPE using the standard massively-parallel pseudo-spectral code FROST [27] with a fourth-order exponential Runge–Kutta temporal scheme (see [17]). We adimensionalize the GPE by measuring length in units of β/α , time in units of β^2/α^3 , and ψ in units of $(\alpha/\beta)^{3/2}$, which amounts to setting $\alpha = \beta = 1$ in equation (1). We set $L = 2\pi$, which means that the nondimensional grid spacing in Fourier space is $\Delta k = 1$, and use grids of N_p^3 collocation points, with $N_p = 512$. We add a forcing term $F_{\mathbf{k}}(t)$ and a dissipation term $-D_{\mathbf{k}}\hat{\psi}_{\mathbf{k}}(t)$ to the RHS of the Fourier-space GPE (6). To obtain an inverse cascade, the forcing term is supported on a narrow band at high k 's around the forcing wavenumber k_f for $k \in [k_f - 1, k_f + 1]$, and it is given by $dF_{\mathbf{k}}(t) = f_0 dW_{\mathbf{k}}$, where $W_{\mathbf{k}}$ is the Wiener process, and the parameter f_0 controls the amplitude of the forcing. Dissipation is implemented via hypoviscosity at the largest scales ($0 < k \leq k_L \ll k_f$), taking the form $D_{\mathbf{k}} = D_L k^{-2r}$, and via hyperviscosity at small scales (high wavenumbers) beyond the forcing range ($k_f + 1 < k \leq k_{\text{max}}$), where $D_{\mathbf{k}} = D_R k^{2s}$. Additionally, the condensate mode ($k=0$) is dissipated at a constant rate D_0 . We optimize the parameters of forcing and dissipation to enlarge the inertial range for a fixed resolution, ensuring that the simulations are well-resolved and minimizing bottlenecks at the dissipation scales. All the numerical parameters are listed in appendix A. We perform simulations with different values of the



forcing strength f_0 in order to study the transition from the weak to the strong WT regimes. The k -space particle flux $Q(k)$ is computed directly from the RHS of the GPE (1) as in [17] (details of this calculation method are provided in appendix B). We evaluate Q_0 (the rate of particle loss per unit volume and unit time through low- k dissipation) by measuring the value of $Q(k)$ in the inertial range where it is approximately constant. Note that f_0 and $|Q_0|$ are monotonically related, so we discuss our results in terms of the strength of the particle flux. Finally, our numerical measurements are performed when a statistically steady state is reached.

Figure 1(a) shows the stationary waveaction spectrum $n(k)$ for values of the flux Q_0 varying over almost five orders of magnitude. The respective spectra compensated by the KZ prediction (5) and the wavenumber compensated by the healing length $\xi = \sqrt{\alpha}/\sqrt{\beta N}$ are shown in figure 1(b). As noted in the numerical methods, we set $\alpha = \beta = 1$ throughout; for clarity, these parameters are omitted from the figure axes and labels hereafter. We observe that for lower values of $|Q_0|$ there is almost perfect agreement with the WWTT KZ spectrum prediction (5)—with both the slope and the prefactor constant of this solution confirmed (as previously seen in [17]). Naturally, the range at which this agreement holds shrinks when the flux $|Q_0|$ increases, and the weak nonlinearity assumption fails at $k\xi \lesssim 1$. This is natural because the healing length ξ is defined as the scale where the linear and the nonlinear terms in the GPE are of a similar size. As scales $k\xi \lesssim 1$, the k^{-4} CB scaling (7) appears when the flux is moderately strong such that $k\xi = 1/\xi$ still lies within the inertial range. Note, however, that CB can only be observed as an intermediate state for moderately high fluxes. Thus, CB is not the *ultimate* strong WT regime. Indeed, for the highest particle flux, we observe a very steep spectrum at low k 's and close to a k^{-2} -spectrum at large wavenumbers. We will later relate this behavior to the emergence of a quasi-condensate with a thermal component induced by the forcing. Condensation occurs when the infrared dissipation becomes insufficient for absorbing the particle flux before it reaches the ground state. The flux functions $Q(k)$ for all runs are provided in figure A1 of appendix B. The figure illustrates that the flux at the lowest non-zero modes, $k=1$, is two-to-three orders of magnitude higher in runs with the condensation (cases 9 and 10) than in the rest of the runs where there was no condensation. When the spectrum pileup occurring at low wavenumbers starts exceeding a certain value, the interaction with these wavenumbers becomes dominant and this changes the dynamics to the acoustic type. Namely, the condensate affects, nonlocally in the k -space, the dynamics of waves with higher wavenumbers: the dominant process is now such that one of the four wavenumbers in the interacting quartet belongs to the low- k condensate. This fact invalidates the locality of the four-wave interaction property used by both the four-wave WWTT and the CB theories.

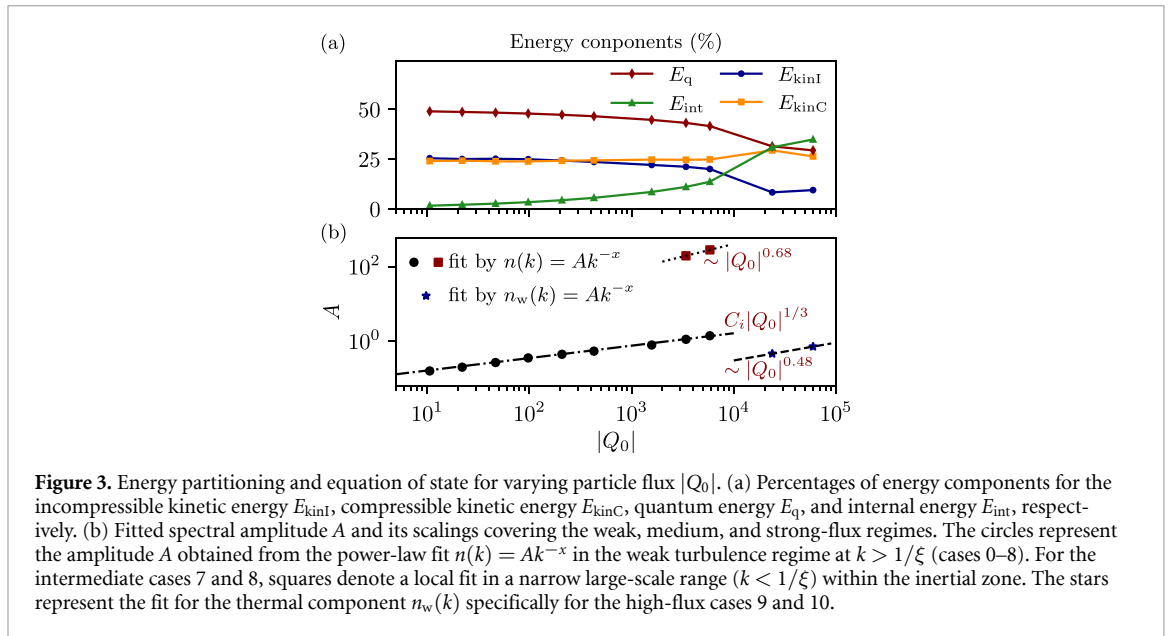


In order to quantitatively study the transition from the WWTT to the strong WT, we make use of the spatio-temporal Fourier transform (STFT) of the wavefunction, which is obtained by performing the Fourier transform in both physical-space coordinate and in time over a finite (but sufficiently large) time window. STFT has become a standard tool in WT systems for checking if the system is in the weakly nonlinear regime: in this case STFT is narrowly concentrated in the proximity of the linear-wave dispersion relation $\omega_k = \alpha k^2$, i.e. the width of the frequency broadening of STFT spectrum at a fixed k , $\delta\omega(k)$, must remain much less than ω_k . Moreover, the CB and strong WT regimes are characterized by $\omega_k \sim \delta\omega_k$ and $\omega_k < \delta\omega_k$, respectively.

Representative examples of the STFT spectral plots are shown in figures 2(a)–(c) and the measurements of the nonlinear frequency broadening $\delta\omega(k)$ (defined here as the width corresponding to 80% of the area under the curve) are in figure 1(d). We can see that in both the weak (a) and the intermediate flux cases (b), the spectrum is distributed closely to the dispersion relation $\omega = \alpha k^2 + 2\beta N$, with $2\beta N$ being the nonlinear shift. The nonlinear broadening displayed in figure 2(d) shows that for the lowest flux case the weak nonlinearity condition for applicability of weak WT, $\delta\omega(k) \ll \omega_k$ is realized over a broad inertial range of wavenumbers between the large-scale dissipation and the forcing scales, $3 \lesssim k \lesssim 100$, the range where the inverse cascade KZ prediction is clearly observed. For the intermediate flux case, the range where $\delta\omega(k) \ll \omega_k$ is also present, but now in a narrower range, $30 \lesssim k \lesssim 100$. KZ spectrum is still seen in this range. On the left of it, in the range $10 \lesssim k \lesssim 30$, we observe $\delta\omega(k) \sim \omega_k$, which is, by definition, the CB state. Accordingly, the CB spectrum is indeed realised in this range. The transition between the KZ and the CB scaling occurs approximately at the healing length wavenumber, $k_\xi \sim 30$. The strong WT case deserves more explanation.

4. Emergence of condensate and acoustic turbulence

When the flux is very intense, the CB assumption is no longer fulfilled and the system transitions to a new regime of strong WT, exhibiting a very steep spectrum at low k 's (figure 1). For this case, the STFT in figure 3(c) displays a behavior completely different from a broadened $\omega_k = \alpha k^2 + 2\beta N$ curve. This state is characterized by the emergence of a strong coherent condensate component around $k=0$. The condensate affects, nonlocally in the k -space, the dynamics of waves with higher wavenumbers, thereby



invalidating the locality property used by both the WWTT and the CB theories. The effect of the condensate takes the form of making the wave perturbations behave as sound characterized by the well-known Bogoliubov dispersion relation,

$$\omega_B(k) = \beta|\psi_0|^2 \pm \sqrt{2\alpha\beta|\psi_0|^2k^2 + \alpha^2k^4}, \quad (8)$$

where $|\psi_0|^2 = |\hat{\psi}_{\mathbf{k}=\mathbf{0}}|^2$ is the condensate density (which is $\approx N$ for a strong condensate). The Bogoliubov dispersion relation arises as a solvability condition for the equations resulting from the linearization of the GPE (1) with respect to a weak wave with wavenumber \mathbf{k} . This dispersion relation is clearly observed in the STFT in figure 3(c), including the nonlinear frequency gap. Weaker branches on the bottom of figure 3(c) appear because $\psi_{\mathbf{k}}$ is a linear combination of the normal-mode amplitudes of the Bogoliubov modes \mathbf{k} and $-\mathbf{k}$.)

Like in the CB regime, the Fjørtoft dual cascade argument does not work in this regime. In the acoustic regime, there is no local inverse cascade, and most of N -flux is now toward the high k 's, whereas the low- k range distribution is necessarily close to thermodynamic equilibrium (i.e. the role of the IR dissipation is negligible). The classical picture for such equilibrium is a condensate distribution $N_0\delta(\mathbf{k})$ superimposed with thermal fluctuations in the form of the energy equipartition of the modes, the so-called Bogoliubov spectrum $n_B(k) \propto T/k^2$ with constant T , which has the meaning of temperature. In an out-of-equilibrium three-wave regime (such as in the presence of a condensate), the flux Q is proportional to the second power of n_k , and thus $T \sim |Q_0|^{1/2}$, which gives

$$n_B(k) \propto \frac{|Q_0|^{1/2}}{k^2}. \quad (9)$$

Note that the Bogoliubov k -scaling (9) coincides with the Rayleigh–Jeans energy equipartition scaling for the weakly interacting Bose gas governed by the four-wave mixing process. Finally, in a finite box, the condensate distribution takes the form of a ‘widened’ δ -function, i.e. a distribution sharply peaked around $k=0$ and rapidly decaying for $k \neq 0$.

The presence of such a condensate corresponds to the short horizontal line at $\omega \approx 1.34\beta N$ in the STFT of figure 2(c), and enhanced in the inset of panel (e). Note that for pure uniform condensate, the STFT spectrum would be concentrated at a point $(k, \omega) = (0, \beta N)$. In our case the condensate spectrum is spread in k , i.e. the condensate is non-uniform in physical space. Also, its frequency is shifted upward with respect to the uniform condensate case. Both of these features can be linked to the finite size of the computational box. One can take advantage of the fact that the wave and the condensate components in the large flux case are clearly separated on the STFT plot, and calculate the spectra of the waves and the condensate separately. To do that we band-pass filter the frequency spectrum in a narrow range $1.30\beta N < \omega < 1.38\beta N$ and call it condensate, while the rest of the spectrum we call waves.

Then, performing the inverse Fourier transform in ω we get the k -spectra shown in figure 2(d), showing a thermodynamic Bogoliubov spectrum of waves over an extended k -range and a very peaked low- k condensate's spectrum decaying approximately as k^{-7} .

Let us clarify the role of vortices in our system for different forcing intensities. Vortices are associated with incompressible kinetic energy, which arises from the Helmholtz decomposition of the regularized velocity field into solenoidal and potential parts. Figure 3(a) displays the energy components for various values of $|Q_0|$: incompressible and compressible kinetic energies, internal energy, and quantum energy. For low $|Q_0|$, quantum energy is dominant and internal energy is minimal, consistent with weak nonlinearity. Compressible and incompressible energies are equal, and each is half the quantum energy. The presence of incompressible kinetic energy indicates vortices, but these are 'ghost' vortices without dynamical significance—they appear even in purely linear fields [28]. Interestingly, increasing $|Q_0|$ reduces incompressible kinetic energy to about 6%–8% in the strongest forcing cases, reflecting a sparse distribution of small vortex loops (with radii of the order of the healing length). This state contrasts with strong superfluid hydrodynamic turbulence states consisting of polarized tangles of fully formed quantized vortex lines of all sizes [29, 30]. In our system, strong vortex lines shrink and annihilate due to effective friction from the intense acoustic component. Persistent vortex tangles would require sound damping, i.e. an effective 'cooling' of the system (recall that the acoustic phonons correspond to a normal fluid component appearing at finite temperature). Thus, the strong turbulence state reported here is dominated by interacting acoustic waves rather than dynamically significant hydrodynamic vortices. A similar annihilation of vortices and transition to an acoustic regime was previously observed in 2D GPE simulations [31].

We will now discuss the out-of-EoS [20, 21], which relates the global amplitude of the waveaction spectrum to the input flux, in our case $|Q_0|$. In practice, we fit our data with $n_k = Ak^{-x}$ in the different observed ranges. The exponents x were found to be very close to those discussed earlier, and the amplitude A is strongly dependent on Q_0 . Readers can find the fitting method and fitted exponents x in appendix C. Figure 3(b) shows the plot of the spectrum's amplitude A vs. the particle flux in the inertial range $|Q_0|$ for our runs covering the whole range of particle fluxes, from low to high. For low-flux runs, we observe $A \sim |Q_0|^{1/3}$ in agreement with the weak four-wave WT. For intermediate fluxes, the spectrum at the high- k part remains weak and still follows $A \sim |Q_0|^{1/3}$. The low- k part, which exhibits the CB exponent, displays a dependence on the flux that could be fitted by $A \sim |Q_0|^{2/3}$. In other words, we obtain $\gamma = 2/3$ in the expression for the CB spectrum (7). Such an exponent evokes some type of hydrodynamic Kolmogorov turbulence, but we lack a theoretical explanation and scaling range is too narrow for considering this exponent's value reliable. Finally, for high-flux runs, we only measure A for the large- k thermodynamic part because the low- k condensate spectrum range is too short for a reliable fitting. Here we see $A = T \sim |Q_0|^{1/2}$, which fulfills our prediction for the temperature scaling of the three-wave process.

5. Conclusions

In this Letter, we studied the stationary inverse cascade states in BEC turbulence for a large range of forcing strengths (or equivalently particle fluxes). We showed that the predictions of the WWTT theory are very robust; namely, the KZ spectrum scaling is observed for fluxes spanning over more than four orders of magnitude. At stronger fluxes, the waveaction spectrum and the STFT spectrum indicate the presence of the CB state in which the nonlinear frequency broadening $\delta\omega(k)$ is close to the linear wave frequency ω_k in a range of wavenumbers k . In the simulations with the strongest flux, we observe states close to low-temperature thermodynamic equilibria, with a prominent condensate component sharply peaked near $k=0$ and over-condensate fluctuations with spectra close to the thermal Bogoliubov spectrum $n_k \sim 1/k^2$. Our complete set of simulations has allowed us to determine a rich equation of state for the 3D inverse cascade BEC turbulence. Experimentally, such a steady-state inverse cascade is yet to be implemented for 3D BEC turbulence. As with a similar setup for the direct cascade [11], the steady state will require simultaneous presence of forcing and dissipation. However, now the dissipation has to withdraw low-momentum particles, and the forcing at high k would have to provide new particles, which is not achievable by shaking the trap (a forcing used in [11]). This could be done using a particle recirculation protocol devised in a recent experiment [14]: it provides effective dissipation at low- k modes through particle removal, while simultaneously acting as a forcing term for particles at small scales by reinjecting them at the forcing scale k_f . Counting the number of removed and reinjected particles provides a natural measurement of the particle flux Q . Clearly, experimental dissipation could not follow the exact shape used in our simulations, but it is likely that the details of the dissipation procedure are

not crucial. The situation is similar with the synthetic dissipation implemented at high k [11], which was different from what is normally considered in the numerics of the direct cascade, but the results confirmed the theory in the inertial (non-dissipative) range of scales. This is consistent with the principle of turbulence universality, which states that the details of the forcing and dissipation are unimportant in the inertial range—it is only the spectral flux Q that matters. Thus, we expect that the essential properties of the steady state will also be observed experimentally in the forced-dissipated inverse cascade settings.

Acknowledgments

This work was funded by the Simons Foundation Collaboration grant WT (Award ID 651471). This work was provided with computing HPC and storage resources of IDRIS and CINES under the allocations A0152A14637 and A0172A14637 made by GENCI on the Jean Zay (SKL partition) and Adastral (GENOA partition) supercomputers. This work was granted access to the OPAL infrastructure from Université Côte d'Azur, supported by the French government, through the UCAJEDI Investments in the Future project managed by the National Research Agency (ANR) under Reference No. ANR-15-IDEX-01.

Data availability statement

The raw simulation data are too voluminous to be hosted on public repositories. The raw data that support the findings of this study are available upon reasonable request from the corresponding author. The source data used to generate all the figures in this study are openly available in the Figshare repository at the following URL: <https://doi.org/10.6084/m9.figshare.31230085> [32].

Appendix A. Numerical parameters

All essential numerical parameters for simulation runs are given in table A1. To systematically study the transition from weak WT to strong turbulence, we fix the size of the computing domain in Fourier space with $k_{\max} = 173$, and the forcing position with $k_f = 125$ for all the runs. The hyperviscous prefactor D_R and exponent s are chosen to absorb the particle flux beyond the forcing range, preventing high- k pileup while avoiding dissipating the inertial range of the inverse cascade. Using hyperviscosity in this way is typical for turbulence simulations to address the finite maximum wavenumber in the simulation, a constraint not present in the original physical system. Similarly, D_L , r and D_0 are chosen to maximize the k -space inertial range by providing controlled dissipation at the largest scales.

Table A1. Parameters for GPE simulations.

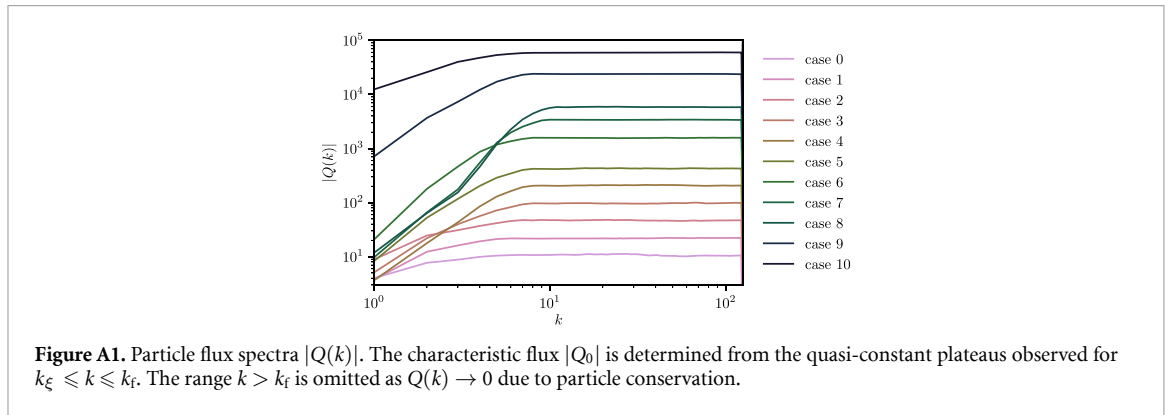
Case	f_0	D_L	r	k_L	D_R	s	D_0
0	10^{-4}	1	0.5	6	4.29×10^{-26}	6	10^3
1	2×10^{-4}	2	0.5	6	2.54×10^{-30}	7	10^3
2	4×10^{-4}	2	0.25	7	1.50×10^{-34}	8	10^3
3	8×10^{-4}	4	0.25	8	8.89×10^{-39}	9	10^3
4	1.6×10^{-3}	8	0.25	8	5.26×10^{-43}	10	10^3
5	3.2×10^{-3}	10	0.25	8	3.11×10^{-47}	11	10^3
6	1.2×10^{-2}	16	0.25	8	1.63×10^{-55}	13	10^3
7	2.5×10^{-2}	70	0.5	10	1.63×10^{-55}	13	10^3
8	5×10^{-2}	250	0.75	11	9.75×10^{-63}	15	10^4
9	0.2	16	0.125	8	9.75×10^{-63}	15	10^4
10	0.5	20	0.125	8	9.75×10^{-63}	15	10^4

Appendix B. Flux spectra

The particle flux spectrum $Q(k)$ is computed as

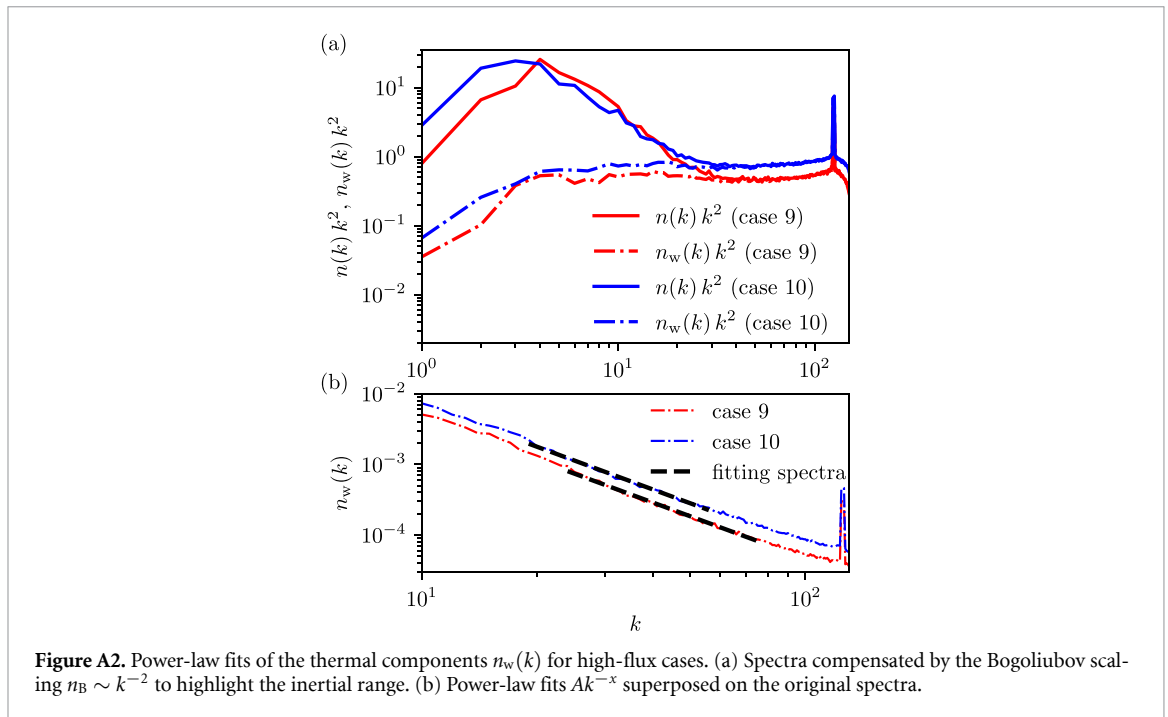
$$Q(k) = -2 \sum_{p=0}^k E_{\psi, \dot{\psi}}(p), \quad (\text{A1})$$

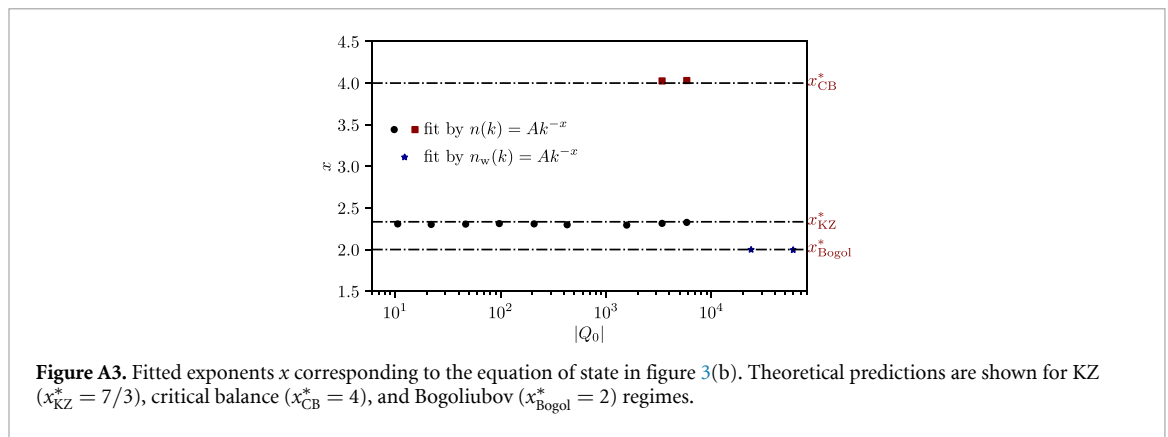
where $E_{f,g}(k) = \sum_{k-\Delta_k/2 \leq |k| < k+\Delta_k/2} \hat{f}_k \hat{g}_k^*$. and $\hat{\psi} = i[\nabla^2 - |\psi|^2] \psi$ the Hamiltonian part on the RHS of the dimensionless GPE. Figure A1 shows the flux spectra $|Q(k)|$ for all runs for $k \leq k_f$. The characteristic steady-state flux magnitude $|Q_0|$ is extracted from the well-defined plateaus in the range $k_\xi \leq k \leq k_f$ and serves as the primary control parameter for the equation of state discussed in the main text.



Appendix C. Method to generate the EoS figure

The out-of-equilibrium EoS is constructed by fitting numerical spectra to the power-law form $n = Ak^{-x}$. For high-flux cases 9 and 10, we fit the thermal wave component $n_w(k)$, whereas for cases 0–8, the total spectrum $n(k)$ is used. This procedure is illustrated in figure A2 for cases 9 and 10, where compensated spectra highlight quasi-constant plateaus in the $n_w(k)k^2$ scaling. We extract the amplitude A and exponent x via linear regression of the logarithmic values, $\ln n = \ln A - x \ln k$, as shown in figure A2(b). This methodology is applied consistently to the weak turbulence ($k > 1/\xi$) and CB ($k < 1/\xi$) regimes. The resulting exponents x are summarized in figure A3 and correspond to the EoS presented in figure 3(b).





ORCID iDs

Ying Zhu  0000-0002-4387-597X

Giorgio Krstulovic  0000-0002-9934-6292

Sergey Nazarenko  0000-0002-8614-4907

References

- [1] Frisch U 1995 *Turbulence: The Legacy of A.N. Kolmogorov* 1st edn (Cambridge University Press) (<https://doi.org/10.1017/CBO9781139170666>)
- [2] Galtier S, Nazarenko S V, Newell A C and Pouquet A 2000 *J. Plasma Phys.* **63** 447–88
- [3] Galtier S 2003 *Phys. Rev. E* **68** 015301
- [4] Lvov Y V and Tabak E G 2001 *Phys. Rev. Lett.* **87** 168501
- [5] Falcon E and Mordant N 2022 *Annu. Rev. Fluid Mech.* **54** 1
- [6] Nazarenko S 2011 *Wave Turbulence* vol 825 (Springer) (<https://doi.org/10.1007/978-3-642-15942-8>)
- [7] Henn E A L, Seman J A, Roati G, Magalhães K M F and Bagnato V S 2009 *Phys. Rev. Lett.* **103** 045301
- [8] Seman J A *et al* 2011 *Laser Phys. Lett.* **8** 691
- [9] Kwon W J, Moon G, Choi J-Y, Seo S W and Shin Y-I 2014 *Phys. Rev. A* **90** 063627
- [10] Navon N, Gaunt A L, Smith R P and Hadzibabic Z 2016 *Nature* **539** 72
- [11] Navon N, Eigen C, Zhang J, Lopes R, Gaunt A L, Fujimoto K, Tsubota M, Smith R P and Hadzibabic Z 2019 *Science* **366** 382
- [12] Madeira L, Caracanhas M, dos Santos F and Bagnato V 2020 *Annu. Rev. Condens. Matter Phys.* **11** 37
- [13] Gałka M, Christodoulou P, Gazo M, Karailiev A, Dogra N, Schmitt J and Hadzibabic Z 2022 *Phys. Rev. Lett.* **129** 190402
- [14] Karailiev A, Gazo M, Gałka M, Eigen C, Satoor T and Hadzibabic Z 2024 *Phys. Rev. Lett.* **133** 243402
- [15] Johnstone S P, Groszek A J, Starkey P T, Billington C J, Simula T P and Helmerston K 2019 *Science* **364** 1267
- [16] Gauthier G, Reeves M T, Yu X, Bradley A S, Baker M A, Bell T A, Rubinsztein-Dunlop H, Davis M J and Neely T W 2019 *Science* **364** 1264
- [17] Zhu Y, Semisalov B, Krstulovic G and Nazarenko S 2023 *Phys. Rev. Lett.* **130** 133001
- [18] Pitajevskij L and Stringari S 2003 *Bose–Einstein Condensation (International Series of Monographs on Physics)* (Clarendon) (available at: <https://books.google.es/books?id=rIobbOxC4j4C>)
- [19] Moreno-Armijos M, Fritsch A, García-Orozco A, Sab S, Telles G, Zhu Y, Madeira L, Nazarenko S, Yukalov V and Bagnato V 2025 *Phys. Rev. Lett.* **134** 023401
- [20] Dogra L H, Martirosyan G, Hilker T A, Glidden J A, Etrych J, Cao A, Eigen C, Smith R P and Hadzibabic Z 2023 *Nature* **620** 521
- [21] Zhu Y, Krstulovic G and Nazarenko S 2024 arXiv:2408.15163
- [22] Goldreich P and Sridhar S 1995 *Astrophys. J.* **438** 763
- [23] Nazarenko S V and Schekochihin A A 2011 *J. Fluid Mech.* **677** 134–53
- [24] Phillips O M 1958 *J. Fluid Mech.* **4** 426–34
- [25] Phillips O M 1985 *J. Fluid Mech.* **156** 505–31
- [26] Proment D, Nazarenko S and Onorato M 2009 *Phys. Rev. A* **80** 051603
- [27] Krstulovic G 2020 A theoretical description of vortex dynamics in superfluids. Kelvin waves, reconnections and particle-vortex interaction *Habilitation à Diriger des Recherches* Université Côte d’Azur (available at: <https://gkrstulovic.gitlab.io/theses/hdr-krstulovic/>)
- [28] Berry M and Dennis M 2000 *Proc. R. Soc. A* **456** 2059
- [29] Polanco J I, Müller N P and Krstulovic G 2021 *Nat. Commun.* **12** 7090
- [30] Fischer T Z and Bradley A S 2025 *Phys. Rev. A* **111** 023308
- [31] Nazarenko S and Onorato M 2006 *Physica D* **219** 1
- [32] Zhu Y, Krstulovic G and Nazarenko S 2026 Data for the paper “Strong and weak wave turbulence regimes in Bose-Einstein condensates.” *figshare* (<https://doi.org/10.6084/m9.figshare.31230085>)