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Received: 19 January 2026

Accepted: 24 June 2026

Published online: 08 July 2026

Cite this article as: Zhu, Y., Krstulovic, G., Nazarenko, S. Turbulence and far-from-equilibrium equation of state of Bogoliubov waves in Bose-Einstein condensates. *Commun Phys* (2026). <https://doi.org/10.1038/s42005-026-02740-2>

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Turbulence and far-from-equilibrium equation of state of Bogoliubov waves in Bose-Einstein condensates

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Abstract

Bogoliubov waves are fundamental excitations of Bose-Einstein condensates. They emerge from a perturbed ground state and interact nonlinearly, triggering turbulent cascades. Here, we study turbulent Bose-Einstein condensates theoretically and numerically using the three-dimensional Gross-Pitaevskii model and its associated wave-kinetic equations. We derive a Kolmogorov-like stationary spectrum for short Bogoliubov waves and find a complete analytical expression for the spectrum in the long-wave acoustic regime. We then use our predictions to explain the far-from-equilibrium equation of state observed in recent turbulent quantum gas experiments, and to suggest future experimental settings.

1 Introduction

Turbulence, in the broad sense, is perhaps one of the most striking manifestations of far-from-equilibrium physics in Nature. In classical fluids, it emerges when the forcing scale, at which energy is injected, is well-separated from the one at which it is dissipated by molecular viscosity. Energy is then carried along scales through a turbulent cascade driven by hydrodynamic interactions. An analogous process occurs in nonlinear wave systems, where cascades are driven by the interaction of waves having different wavelengths. When the wave nonlinearity is weak, the Wave Turbulence Theory (WTT) provides a comprehensive theoretical framework describing far-from-equilibrium steady and evolving states [1]. The theory has

been successful in describing, for instance, the dynamics of gravitational waves [2], Kelvin waves on superfluid vortices [3], internal gravity waves in the ocean [4, 5], Langmuir waves in plasmas [6] and, more importantly for this work, matter waves in turbulent Bose-Einstein condensates (BECs) [7, 8].

One of the most interesting and important regimes in BEC turbulence is realised when weak waves propagate on a background of a strong coherent ground state Ψ_0 , the condensate. These excitations are the Bogoliubov waves, whose frequency ω is given by the dispersion relation,

$$\omega_k \equiv \omega(k) = c_s k \sqrt{1 + (k\xi)^2/2}, \quad (1)$$

where $k = |\mathbf{k}|$ is the modulus of the wave vector $\mathbf{k}$, c_s is the speed of sound, and ξ is the healing length of the BEC at which dispersive effects become important. The wave field $\Psi(\mathbf{x}, t)$ can be viewed as a ground condensate with superimposed perturbations that $\Psi(\mathbf{x}, t) = \Psi_0(\mathbf{x}, t) + \delta\Psi(\mathbf{x}, t)$. Bogoliubov waves emerge when the condensate amplitude $|\Psi_0|$ is significantly larger than the perturbations $|\delta\Psi|$, leading to quadratic nonlinearity and 3-wave interactions.

Over the last few years, outstanding progress has been achieved in realising turbulent BECs experimentally, and they have become an excellent platform for studying far-from-equilibrium physics. Turbulent states are driven by external forcing and dissipation, which results in an energy flux P_0 through scales (which equals the energy injection and dissipation rates). Such fluxes thus play a role equivalent to that of temperature in equilibrium thermal states. It is then natural to ask if there is a universal far-from-equilibrium equation of state (EoS) that relates energy flux and internal observables, such as energy or particle momentum distribution. The clean and well-controlled turbulent BEC experiments available today [9], together with recent numerical simulations of the Gross-Pitaevskii equation [10], have led to revealing the existence of such EoS. They have shown that the amplitude of the wave-action spectrum scales with the flux as $P_0^{1/3}$, followed by a $P_0^{2/3}$ scaling for large flux. Whereas the $P_0^{1/3}$ -scaling can be understood within the WTT invoking 4-wave interactions, the second scaling-law lacks explanation.

In the first part of the results, we obtain theoretical predictions for 3D BEC turbulence dominated by interacting Bogoliubov waves, which are then confirmed by our numerical simulations. Among the most significant outcomes of our work are the following stationary energy spectra of Kolmogorov-Zakharov (KZ) type, which include the universal constants, for the acoustic and short wave limits, respectively:

$$E(k) = C_1 c_s^{1/2} P_0^{1/2} k^{-3/2}, \quad \text{for } k\xi \ll 1, \quad (2)$$

$$E(k) = C_2 c_s^{1/2} \xi^{5/2} P_0^{1/2} k, \quad \text{for } k\xi \gg 1, \quad (3)$$

where the universal constants are derived as $C_1 = 1/\sqrt{3\pi(\pi + 4\ln 2 - 1)}$ and $C_2 =$

$2^{3/4}/\sqrt{\pi(\pi - 4\ln 2)}$. As an application of our theory, in the second part of the results, we provide a plausible theoretical explanation of the second scaling law of the EoS observed experimentally in [9]. This experiment used a cylinder-shaped box trap that was shaken to excite turbulence and which had a "finite height" thereby providing a dissipation (escape) of high-momentum particles. We re-analyse the experimental data of [9], and show their agreement with our prediction of scaling $P_0^{1/2}$ arising from interacting short-wave Bogoliubov modes. Our findings are important for BECs in a broad context, and as they are universal and expected to be generically realisable in turbulent systems, their impact thus goes beyond the explanation of a particular experiment.

2 Results and Discussion

2.1 Stationary Kolmogorov-type spectra for Bogoliubov waves

The master model for describing BECs is the Gross-Pitaevskii equation (GPE) for the complex scalar wave function $\Psi(\mathbf{x}, t)$, where $\mathbf{x}$ is the position in 3D physical space and t is time. Defining Ψ such that $|\Psi|^2$ is the mass density, the GPE, written in terms of ξ , c_s and the unperturbed (uniform) condensate mass density ρ_0 , reads

$$i \frac{\partial \Psi}{\partial t} = \frac{c_s}{\sqrt{2}\xi} \left[-\xi^2 \nabla^2 + \frac{|\Psi|^2}{\rho_0} - 1 \right] \Psi + V(\mathbf{x}, t) \Psi + i \mathcal{F} - i \mathcal{D}. \quad (4)$$

We have also included an external potential trap $V(\mathbf{x})$, a large-scale forcing $\mathcal{F}$ and a hyper-viscous dissipation term $\mathcal{D}$ acting at small scales. In BEC experiments, dissipation is achieved synthetically by allowing high energy particles to escape from the trap thereby removing energy from the system. In our simulations we mimic this effect by the hyper-viscous term which effectively acts on the high- k particles only. Moreover, in experiments, energy injection is provided by shaking the confining trap. First, in our idealized simulations, we set $V \equiv 0$, but we include an external forcing in order to obtain a non-equilibrium steady state. Then, we verify our predictions in a trapped BEC, with an oscillating trap (and $\mathcal{F} = 0$). The healing length and the speed of sound depend both on the physical properties of the BEC and on ρ_0 .

The GPE can be mapped to an effective compressible irrotational fluid flow via the so-called Madelung transformation, $\Psi(\mathbf{x}, t) = \sqrt{\rho(\mathbf{x}, t)} \exp[i\phi(\mathbf{x}, t)/\sqrt{2}c_s\xi]$ with $\rho(\mathbf{x}, t)$ and $\phi(\mathbf{x}, t)$ being the fluid mass density and the velocity potential, respectively. The Bogoliubov dispersion relation Eq. (1) is obtained by linearizing the GPE, Eq. (4), in the absence of the potential, forcing, and dissipation terms, considering perturbations over the ground state, $\delta\Psi = \Psi - \sqrt{\rho_0}$, and diagonalizing the linearized equations in Fourier space.

In our theory, we consider scales much smaller than the size of the potential trap, which allows us to consider the large-box limit and $V(\mathbf{x}) \equiv 0$. Then, for weakly nonlinear waves, the leading order interaction occurs between the modes satisfying 3-wave resonant conditions,

$$\omega_k = \omega_{k_1} + \omega_{k_2}, \quad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2. \quad (5)$$

In this limit, the WTT provides a wave-kinetic equation (WKE) describing the evolution of the wave-action spectrum $n_{\mathbf{k}}^B = n^B(\mathbf{k}, t)$ which is the Bogoliubov quasi-particle momentum distribution defined in Eq. (17) in Methods. The general WKE for weakly nonlinear waves driven by 3-wave resonant interactions is [1, 11]

$$\frac{\partial n_{\mathbf{k}}^B}{\partial t} = \text{St}_{\mathbf{k}}, \quad (6)$$

with the wave-collision integral

$$\begin{aligned} \text{St}_{\mathbf{k}} &= 2\pi \int (\mathcal{R}_{12}^{\mathbf{k}} - \mathcal{R}_{k_2}^1 - \mathcal{R}_{k_1}^2) d\mathbf{k}_1 d\mathbf{k}_2, \\ \mathcal{R}_{12}^{\mathbf{k}} &= \delta_{12}^{\mathbf{k}} \delta(\omega_{12}^k) |V_{12}^k|^2 [n_{\mathbf{k}_1}^B n_{\mathbf{k}_2}^B - n_{\mathbf{k}}^B n_{\mathbf{k}_1}^B - n_{\mathbf{k}}^B n_{\mathbf{k}_2}^B], \\ \delta_{12}^{\mathbf{k}} &= \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2), \\ \delta(\omega_{12}^k) &= \delta(\omega_k - \omega_{k_1} - \omega_{k_2}), \end{aligned} \quad (7)$$

where $V_{12}^k \equiv V(k, k_1, k_2)$ is the 3-wave interaction amplitude. For Bogoliubov waves

$$\begin{aligned} V_{23}^1 &= V_0 \sqrt{k_1 k_2 k_3} W_{23}^1, \quad V_0 = \frac{3\sqrt{c_s}}{4\sqrt{2}}, \\ W_{23}^1 &= \frac{1}{2\sqrt{\eta_1 \eta_2 \eta_3}} + \frac{\sqrt{\eta_1 \eta_2 \eta_3}}{6k_1 k_2 k_3} \left(\frac{k_1^3}{\eta_1} - \frac{k_2^3}{\eta_2} - \frac{k_3^3}{\eta_3} \right), \end{aligned} \quad (8)$$

where $\eta_i \equiv \eta(k_i) = \sqrt{1 + (k_i \xi)^2/2}$ for $i = 1, 2, 3$. The Bogoliubov WKE Eq. (8) was first derived in [7] in a slightly different form with mistakes in the prefactor. Finally, by assuming isotropy and performing an angle average, the WKE becomes

$$\begin{aligned} \frac{\partial n_k^B}{\partial t} &= \text{St}_k = 4\pi^2 V_0^2 \int_{k_1, k_2 \geq 0} (|W_{12}^k|^2 \mathcal{T}_{12}^k \delta(\omega_{12}^k) \\ &\quad - |W_{k_2}^1|^2 \mathcal{T}_{k_2}^1 \delta(\omega_{k_2}^1) - |W_{k_1}^2|^2 \mathcal{T}_{k_1}^2 \delta(\omega_{k_1}^2)) k_1^2 k_2^2 dk_1 dk_2, \end{aligned} \quad (9)$$

where $\mathcal{T}_{12}^k = n_{k_1}^B n_{k_2}^B - n_{\mathbf{k}}^B n_{\mathbf{k}_1}^B - n_{\mathbf{k}}^B n_{\mathbf{k}_2}^B$ (Methods). Note that only resonant waves interact.

Acoustic limit. — In the large-scale limit $k_i \xi \rightarrow 0$, we have $\eta_i \rightarrow 1$ and therefore $\omega_{k_i} = c_s k_i$ — this is the acoustic-wave limit. Also, on the resonant manifold Eq. (5), we have $W_{23}^1 \rightarrow 1$, and thus $V_{23}^1 \rightarrow V_0 \sqrt{k_1 k_2 k_3}$, consistently with the result in [12]. The collisional integral of the WKE Eq. (9) becomes as in the 3D acoustic WKE [13],

$$\text{St}_k = \frac{4\pi^2 V_0^2}{c_s} \int_{k_1, k_2 \geq 0} (\mathcal{T}_{12}^k \delta_{12}^k - \mathcal{T}_{k_2}^1 \delta_{k_2}^1 - \mathcal{T}_{k_1}^2 \delta_{k_1}^2) k_1^2 k_2^2 dk_1 dk_2, \quad (10)$$

where $\delta_{12}^k = \delta(k - k_1 - k_2)$.

In [13], Zakharov and Sagdeev obtained a Kolmogorov-type stationary solution for the energy spectrum of 3D acoustic waves: $E(k) = 4\pi k^2 \omega_k n_k^B = C_1 c_s^{1/2} P_0^{1/2} k^{-3/2}$, the so-called Zakharov-Sagdeev (ZS) spectrum. Here, the energy flux P_0 is constant because of the stationarity of the spectrum. An approximation of the dimensionless constant C_1 was given in [13], as well in a recent paper [14]. We derive its exact analytical expression, and obtain the complete ZS spectrum, Eq. (2) (Methods).

Note that the acoustic WKE Eq. (10), is derived as an asymptotic limit of Eq. (9). Unlike the 2D case where $\text{St}_k \propto \xi^{-1}$ [12], in 3D ξ cancels out after angle averaging [11] leading to Eq. (10). However, for its validity, a small amount of dispersion, naturally present in BECs, is still crucial in the system to regularize the wave dynamics and avoid singularities.

Short-wave limit. — In the small-scale limit $k\xi \rightarrow \infty$, we have $\eta_k \rightarrow k\xi/\sqrt{2}$ and $V_{23}^1 \rightarrow$

$2^{7/4}V_0\xi^{-3/2}/3$. The collisional integral becomes

$$\begin{aligned} \text{St}_k = & \frac{32\sqrt{2}\pi^2V_0^2}{9\xi^3k} \int_{k_1, k_2 \geq 0} (\mathcal{T}_{12}^k \delta(\omega_{12}^k) - \mathcal{T}_{k2}^1 \delta(\omega_{k2}^1) \\ & - \mathcal{T}_{k1}^2 \delta(\omega_{k1}^2)) k_1 k_2 dk_1 dk_2. \end{aligned} \quad (11)$$

where now $\omega_k = c\xi k^2/\sqrt{2}$. Note that this collision integral coincides with the one for the finite- k spectral part $n_k^{\text{B}'}$ obtained from the 4-wave kinetic equation via substitution $n_k^{\text{B}} = N_0\delta(\mathbf{k}) + n_k^{\text{B}'}$, $N_0 = \text{const}$; see e.g. [15].

To find the Kolmogorov-type spectrum in the short-wave limit, we apply the standard procedure provided by the WTT and seek a stationary power-law solution of Eq. (11), $n_k^{\text{B}} = Ak^{-x}$, $A = \text{const}$. This leads to $\text{St}_k = 64V_0^2\pi^2A^2(9\xi^4c_s)^{-1}k^{1-2x}I(x)$, with the dimensionless collision integral

$$\begin{aligned} I(x) = & \int_{q_1, q_2 \geq 0} (q_1 q_2)^{1-x} \left((1 - q_1^x - q_2^x) \delta(1 - q_1^2 - q_2^2) \right. \\ & \left. - 2(q_1^x - 1 - q_2^x) \delta(q_1^2 - 1 - q_2^2) \right) dq_1 dq_2. \end{aligned} \quad (12)$$

The standard definition of the energy flux is given by $P(k) = -\int_0^k 4\pi q^2 \omega(q) \text{St}(q) dq = \frac{128\sqrt{2}}{9}\pi^3 V_0^2 \xi^{-3} A^2 \frac{I(x)}{2x-6} k^{6-2x}$. Constant energy flux, $P(k) \equiv P_0 = \text{const}$, implies $2x - 6 = 0$, i.e. $x = 3$ and $I(3) = 0$. Using the L'Hôpital rule, $\lim_{x \rightarrow 3} I(x)/(x - 3) = dI(x)/dx|_{x=3}$. We evaluate this expression and find the steady spectrum, Eq. (3) (Methods). This spectrum, as well as the constant C_1 in Eq. (2), are theoretical predictions derived here. Note that the constants C_1 and C_2 are finite because the respective integrals defining the energy flux are convergent, a property called locality of the wave interactions (Methods). We will now test these predictions numerically via simulations of the GPE and the isotropic WKE.

2.2 Numerical validation of the wave turbulence predictions

First, we numerically simulate the forced-dissipated GPE, Eq. (4), setting $\rho_0 = 1$ and $c_s = 1$, in a triply periodic cube with the side length $L = 2\pi$, and $V(\mathbf{x}) \equiv 0$. To focus on different ranges with respect to the transitional wave

number $k_\xi = 1/\xi$, two high-resolution GPE simulations with $\xi = 1.25\Delta x$ and $\xi = 15\Delta x$ (Δx is the length of the grid spacing) are performed. Such choices allow us to study the acoustic and the short-wave limits separately. We set the initial data with uniform condensate with $|\Psi(0)|^2 = \rho_0$, a stochastic forcing then adds the wave disturbances, and we evolve the system until it reaches a steady state, balancing between the large-scale forcing and the small-scale dissipation (Methods). We then perform averages over time. Also, we perform two isotropic WKE simulations: one of the Bogoliubov WKE, Eq. (9), and the other of the purely acoustic WKE, Eq. (10), with extra forcing and dissipation terms similar to the ones used in the GPE simulations (Methods).

The results of these runs in terms of the energy spectra and the fluxes superimposed with the respective theoretical predictions, Eq. (2) and Eq. (3), are presented in Fig. 1. The energy fluxes shown in Fig. 1(b) exhibit plateau ranges, with $P(k) = P_0 = \text{const}$, in which the spectra are stationary and both forcing and dissipation effects are negligible (the so-called inertial ranges). The measured values of P_0 are then used for normalising the energy spectra in Fig. 1(a). We can see that each of the respective inertial ranges, for both WKE (run 1 and run 2) and GPE (run 3 and run 4) simulations, are in excellent agreement with our theoretical predictions (both for the spectral slopes and the prefactor constants). The longest inertial range, spanning for about two decades, is realised in the acoustic WKE run (run 2) with scaling spectrum Eq. (2) confirmed. The Bogoliubov WKE run (run 1) produced two scaling ranges in the large-scale and the short-scale parts of the spectrum simultaneously (about a decade-wide each). The GPE runs with small and big values of ξ produced one scaling each for $k\xi < 1$ (run 3) and $k\xi > 1$ (run 4). The scaling range in the GPE run 3 is the narrowest among all the simulations, but its spectrum agrees with the Bogoliubov WKE simulation. In particular, it exhibits a bump in the high- k part of the spectrum close to the transitional wave number, $k \sim k_\xi$. A similar bump was previously observed in GPE simulations of [16], and it was called a "dispersive bottleneck" since it is related to an increased wave dispersion in this region of scales. Note that our predictions are well confirmed by the GPE simulations without any adjustable parameters.

In BEC turbulence experiments, the cloud is always confined by an external potential trap. To be more realistic, we perform two further GPE simulations in a box-shaped potential trap of size $L_{\text{trap}} = 1.5\pi$, with $\xi = 1.25\Delta x$ (run 5) and $\xi = 10\Delta x$ (run 6), namely $L_{\text{trap}} = 230.4\xi$ and $L_{\text{trap}} = 28.8\xi$, respectively. We prepare ground states so that the initial density inside the trap is nearly constant with $|\Psi(0)|^2 = \rho_0$. After that, we shake the trap in both the x and y directions to introduce energy into the system until the small-scale energy dissipation rate becomes equal to the injection rate. We adjust the shaking protocol and the dissipation parameters to keep the system in the Bogoliubov regime ensuring that at least 70% of the particles are staying at $k \leq k_{\text{trap}} = 2\pi/L_{\text{trap}}$. We measure the energy flux P_0 by calculating the dissipation rate of energy on small scales, in the same spirit as in [9] (Methods). Fig. 1(c) and (d) show the density snapshots for runs 5 and 6, respectively. Although the presence of traps reduces the width of the inertial range, Fig. 1(a) shows that the spectra obtained with the potential trap remain consistent with the WTT-predicted scalings in both regimes (run 5 for $k\xi \sim 0.08 - 0.5$, and run 6 for $k\xi \sim 2 - 8$). On the other hand, in both cases, the interaction with the potential trap increases the effective spectrum constants: by a factor ~ 4 in the acoustic regime and ~ 1.2 for the short waves.

2.3 Short-wave equation of state predicted by four-wave and three-wave theories

Almost all BEC experiments relevant to wave turbulence have been interpreted within the framework of the 4-wave interaction theory [9, 17, 18], which assumes that the wave perturbations $\delta\Psi$ greatly exceed the condensate amplitude Ψ_0 , leading to cubic nonlinearity. Instead, in this work we employ the short-wave solution derived above, Eq. (3) governed by 3-wave interactions to interpret the experimental results of Dogra et al. [9] on the far-from-equilibrium EoS. Dogra et al. analysed the particle spectra (occupation number spectra) n_k (which has a difference of prefactor of $8\pi^3$ from the one we defined as Eq. (18) in Methods) measured in the direct cascade of BEC turbulence by fitting a scaling-law of $k^{-3+\delta}$, with δ being an adjustable parameter. Note that at small

scales, our prediction Eq. (3) corresponds to a k^{-3} wave-action spectrum (equivalent to the particle spectrum, in this case, up to a prefactor difference; see Eq. (20) in Methods), which coincides, up to the log-correction, with the 4-wave prediction found in [8]. This coincidence makes both spectra good candidates for explaining the experimentally found EoS, assuming that they correspond to the short-wave range such that $k\xi > 1$.

The direct cascade solution of the 4-wave WKE found in [19], recast in a dimensional form, reads

$$n_k = n_{4w} k^{-3} \ln^{-\frac{1}{3}} \left(\frac{k}{k_f} \right), \quad n_{4w} = C_{4w} \left(\frac{m^2 \epsilon}{\hbar^3 a_s^2} \right)^{\frac{1}{3}}, \quad (13)$$

where k_f is the forcing wave number and $C_{4w} = C_d(4\pi)^{-\frac{2}{3}}$, with $C_d \approx 5.26 \times 10^{-2}$ being a universal constant. Here, m is the atom's mass, $\hbar$ is the reduced Planck constant, and a_s is the s -wave length. The energy flux ϵ is defined as the energy dissipated in the system per unit time per unit volume.

In [9], the authors performed a series of experiments with various parameters, interaction strength a_s , forcing intensity, and mean particle density N such that $N = \int 4\pi k^2 n_k dk$ with the definition of n_k as in this paper. They obtained the particle spectra amplitude n^{exp} by fitting the experimental spectra with $n_k = n^{\text{exp}} k^{-3.2}$, and discovered that employing the dimensional variables n^{exp} and ϵ as two state variables results in a non-universal EoS for different a_s and N . Furthermore, they empirically found that the dependence of the non-dimensional spectrum amplitude, n_{exp}/N versus the non-dimensional energy flux $\tilde{\epsilon} = m^2 \hbar^{-3} a_s^{-2} N^{-3} \epsilon$ collapses the experimental data into a universal EoS. Theoretical prediction Eq. (13) implies

$$n_{4w}/N = C_{4w} \tilde{\epsilon}^{1/3}, \quad (14)$$

which was confirmed in [9] for data points with low values of $\tilde{\epsilon}$. However, at larger $\tilde{\epsilon}$, [9] reported a steeper EoS, $n_{4w}/N \sim \tilde{\epsilon}^\alpha$ with $\alpha > 1/3$. The authors of [9] noted that $\alpha \approx 2/3$ provides a good fit implying a flux-dependence like the one of the classical Kolmogorov hydrodynamic turbulence. However, it is unclear how a state with the classical Kolmogorov turbulence could appear in this experiment, considering that it is unlikely that

the cascade process is generated by vortices with hydrodynamic properties. Indeed, such regimes are usually attributed to states with polarised vortex tangles at scales greater than the inter-vortex distance [20], which do not seem to be relevant to the considered experiment. Also, as noted in [9], the Kolmogorov spectrum would imply a spectral exponent $-5/3$ that is very different from the observed exponent ≈ -3 .

Here, we suggest a plausible interpretation of the steeper EoS. We propose that a stronger ground-state condensate component could be present for greater $\tilde{\epsilon}$ making the system switch to the 3-wave turbulence of short Bogoliubov waves, implying $\alpha \approx 1/2 > 1/3$.

Recasting our prediction Eq. (3) in the same manner as in Eq. (13), we obtain (Methods)

$$n_k = n_{3w} k^{-3}, \quad \frac{n_{3w}}{N} = C_{3w} \left(\frac{m^2 \epsilon}{\hbar^3 N^3 a_s^2} \right)^{\frac{1}{2}}, \quad (15)$$

where $C_{3w} = \frac{C_2}{2^{17/4} \pi^2}$. The dimensional P_0 in Eq. (2) is related to ϵ as $\epsilon = \rho_0 P_0$, since P_0 complies with the standard definition of the energy flux as the energy dissipation rate per unit mass whereas ϵ used in [9] is the energy dissipation rate per unit volume. The EoS corresponding to Eq. (15) is

$$n_{3w}/N = C_{3w} \tilde{\epsilon}^{1/2}. \quad (16)$$

This EoS looks similar to the 4-wave EoS Eq. (14): both laws are of the form $n/N = C \tilde{\epsilon}^\alpha$. However, one should not forget the important difference between the 4-wave and the 3-wave case: the non-normalised 4-wave EoS given by Eq. (13) does not contain any N -dependence whereas the 3-wave law given in Eq. (15) does involve N .

2.4 Re-analysis of the experimental equation of state

The revised EoS based on the experimental data points of [9] is shown in Fig. 2. Our procedure is different from the one of [9]: for each experimentally obtained spectrum n_k^{exp} , we try our two theoretical predictions, namely $n_k = n_{4w}^{\text{exp}} k^{-3} \ln^{-\frac{1}{3}} \left(\frac{k}{k_f} \right)$, and $n_k = n_{3w}^{\text{exp}} k^{-3}$, investigate the compensated spectra $n_k^{\text{exp}} k^3 \ln^{1/3} \left(\frac{k}{k_f} \right)$ and $n_k^{\text{exp}} k^3$, respectively, and compute n_{4w}^{exp} and n_{3w}^{exp} by averaging the corresponding compensated spectrum in their plateau ranges, as shown in the

inset of Fig. 2(a) (see the Methods for an example where the 4-wave prediction is selected). We take the value of the spectrum amplitude, n^{exp} , either n_{4w}^{exp} or n_{3w}^{exp} , depending on which is closer to its corresponding prediction, n_{4w} or n_{3w} . In the other words, we find the best fit for the compensated spectra in terms of the L^1 -norm. We plot the fitting spectrum amplitude n^{exp} vs. the flux ϵ in Fig. 2(a), magenta for the points taken with n_{4w}^{exp} and green for those taken with n_{3w}^{exp} . The non-dimensional EoS is presented in Fig. 2(b), in which the points split into two distinct clouds and collapse onto a universal EoS. Note that in [9], the authors reduced the spread of their EoS by introducing two extra phenomenological fitting parameters in addition to n^{exp}/N and $\tilde{\epsilon}$. Our fitting procedure is based only on comparison with our two theoretical predictions without introducing any adjustable parameters.

Fig. 2(b) also shows the theoretical 4-wave and 3-wave EoS predictions in which the constants C_{4w} and C_{3w} are replaced by the mean values of the respective constants obtained by fits of the experimental spectra, $\langle C_{4w}^{\text{exp}} \rangle$ and $\langle C_{3w}^{\text{exp}} \rangle$. To provide a visual indication of the fit quality, shaded bands are added around the two fitted curves; they correspond to one standard deviation in logarithmic scale of the data relative to the fitted lines. We see that the theoretical predictions are consistent with the experimental data, with the 4-wave and 3-wave EoS in agreement with the low and high $\tilde{\epsilon}$ parts respectively. The mean experimental constants are nearly twice the theoretically predicted values, $\langle C_{3w}^{\text{exp}} \rangle \approx 2.2 C_{3w}$ and $\langle C_{4w}^{\text{exp}} \rangle \approx 2.4 C_{4w}$. The deviation of the experimentally measured constants from their theoretical predictions is of the same order as the spread (L^∞ -norm, $\Delta C^{\text{exp}} = C_{\text{max}}^{\text{exp}} - C_{\text{min}}^{\text{exp}}$) of the EoS data points from the mean, $\Delta C_{3w}^{\text{exp}} \approx 2.3 C_{3w}$ and $\Delta C_{4w}^{\text{exp}} \approx 1.7 C_{4w}$. Since there are no adjustable parameters in our theoretical results and fitting process, and theoretical predictions are asymptotic for $k\xi \gg 1$ (which in experiments is not satisfied in a strong sense, as $k\xi \approx 4$) we conclude that our theoretical predictions reasonably agree with the experimental data. Our numerical simulations with shaken traps, which more closely correspond to the experimental setup, show that the interaction with the trap tends to increase the effective constants of the stationary KZ-type spectra relative to the corresponding simulations without the trap. This trend

is also consistent with our previous study of [8], and agrees with the observation that $\langle C_{4w}^{\text{exp}} \rangle > C_{4w}$ and $\langle C_{3w}^{\text{exp}} \rangle > C_{3w}$. A possible explanation is that, in the box-trap configuration, the zero boundary conditions replace the plane-wave eigenmodes of the homogeneous case by standing-wave, sine-like modes. This significantly reduces the number of active resonant modes and effectively decreases the prefactor of the WKE collision integral. As a consequence, for a given flux, this tends to produce a larger effective constant of the stationary KZ-type spectrum. This effect is expected to be more pronounced in smaller systems, where the available scale separation is insufficient. We expect that in larger traps, the experimental measurements would become closer to the universal limit and their measured constants would tend towards the predicted values. Nevertheless, further studies of the inhomogeneity induced by traps and respective systematic deviations in numerical (and experimental) flux measurements are desirable to better comprehend the finite size effects.

3 Conclusions

In summary, in this paper we have obtained the Kolmogorov-type stationary spectrum of forced-dissipated turbulence of short Bogoliubov waves, Eq. (3), and we revised the long-wave Kolmogorov-type spectrum including finding the exact prefactor constant C_1 , see Eq. (2). We have validated these solutions, along with the combined spectrum including both long and short waves by numerical simulations of the GPE and the isotropic Bogoliubov WKE. Our work relies on the turbulence universality hypothesis, stating that the inertial (non-dissipative) range spectrum is insensitive to specific forcing and dissipation mechanisms. Consequently, we adopted stochastic forcing and hyperviscous dissipation to maximize the inertial range, as is customary in turbulence simulations. Our predictions were also tested in more realistic settings, by using an oscillating external trap to force the system.

We used the short-wave spectrum derived in this paper for interpreting the (previously unexplained) experimental Equation of State of Ref. [9]. To test our theory in future experiments, the realisability of Bogoliubov-wave turbulence in both asymptotic limits should be further examined. For the short-wave regime, although

the present comparison with [9] already shows a reasonable agreement at $k\xi \sim 4$, a cleaner quantitative verification of the asymptotic prediction would benefit from experiments reaching larger values of $k\xi$. For the long-wave acoustic regime, future verification is expected to require sufficiently large systems, with L_{trap}/ξ of order 100. In the short-wave regime, one should additionally check: (i) whether a strong ground state is present for strong forcing cases, because in the Gross-Pitaevskii model the relevant interaction depends on the background on which the fluctuations propagate: the three-wave description applies when $|\Psi_0| \gg |\delta\Psi|$, whereas the four-wave description applies in the opposite limit $|\Psi_0| \ll |\delta\Psi|$; (ii) whether, for these cases, the measured spectrum corresponds to waves shorter than the healing length and with amplitudes weaker than the ground state. A plausible route to the three-wave regime in the experiment is that, at large fluxes, the turbulent steady state is established before substantial long-time condensate depletion occurs, so that the short-wave fluctuations still propagate on a strong condensate background during the measurement window. At lower fluxes, by contrast, the steady state is established more slowly (see Fig. 2(a) of [9]), and the longer waiting time may allow stronger condensate depletion, thereby favouring the four-wave regime. This interpretation, however, still requires direct experimental verification. It should be emphasised that, within the WTT framework, the three-wave and four-wave regimes correspond to distinct asymptotic limits, namely $|\Psi_0| \gg |\delta\Psi|$ and $|\Psi_0| \ll |\delta\Psi|$, respectively. It is natural to expect that some points in a real experimental system do not fall into either of the two ideal limits, but rather lie in an intermediate range where $|\Psi_0| \sim |\delta\Psi|$. This may contribute to the remaining variance between our predictions and the experimental data. Therefore, our classification of the experimental points in Fig. 2 is meant to identify the dominant scaling behaviour and should be understood as a leading-order interpretation. The transitional regime corresponds to stronger waves whose statistics is no longer described by standard WTT. A future model is anticipated to describe short scales in this transitional regime, where the condensate amplitude and fluctuation amplitude are comparable.

Although the non-dispersive acoustic ZS spectrum is not the main topic of this work, it is an interesting side result that we would like to comment on. We find a way to regularize the WKE of non-dispersive acoustic waves for the 3D isotropic case, but get a different prefactor. Consequently, the ZS spectrum comes with a bigger constant (Methods). In recent work [14], the authors numerically obtained the ZS spectra in 3D acoustic wave systems, and different values of constant were reported for the dispersive and non-dispersive cases. The constant observed in [14] for the non-dispersive acoustic system is smaller than the one for the dispersive system, unlike what we get here. Their paper offers an explanation based on a numerical observation of an anisotropy (i.e. presumably the isotropic spectrum is unstable with respect to anisotropic perturbations). Note, however, that the model acoustic-wave equation studied in [14] was different from the compressible Euler equation or GPE. Thus, the question about the acoustic wave spectrum arising in the non-dispersive Euler equation merits further study.

Methods

Wave-kinetic equations and steady Kolmogorov-type spectrum

The starting point to derive the 3-wave interaction amplitude V_{23}^1 in Eq. (8) is to rewrite the Hamiltonian of the second order H_2 into the canonical form of $H_2 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{k}}^*$, then the wave-action spectrum is defined as

$$n_{\mathbf{k}}^B = \left(\frac{L}{2\pi} \right)^3 \langle |a_{\mathbf{k}}|^2 \rangle, \quad (17)$$

where the problem is considered in an L -periodic box. As a consequence, one rewrites the Hamiltonian of third order into $H_3 = \sum_{1,2,3} V_{2,3}^1 (a_1 a_2^* a_3^* + \text{c.c.}) \delta_{2,3}^1$ and gets the interaction amplitude. In Supplementary Note 1, we provide two methods to derive V_{23}^1 . Just to mention one of the two methods here, we first assume $\Psi(\mathbf{x}, t) = \Psi_0(1 + \psi(\mathbf{x}, t))$, such that $|\Psi_0|^2 = \rho_0$, $\langle \psi(\mathbf{x}, t) \rangle = 0$ and $|\psi| \ll 1$, and define the Fourier series as $\psi(\mathbf{x}) = \sum_{\mathbf{k}} \psi_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}$. We obtain $a_{\mathbf{k}}$ from $\psi_{\mathbf{k}}$ by finding a linear Bogoliubov transformation, and this allows us to find the relationship between the particle spectrum (ignoring the zero mode),

defined as

$$n_{\mathbf{k}} = \left(\frac{L}{2\pi} \right)^3 \langle |\Psi_{\mathbf{k}}|^2 \rangle / m = \left(\frac{L}{2\pi} \right)^3 \langle |\psi_{\mathbf{k}}|^2 \rangle \rho_0 / m, \quad (18)$$

and the wave-action spectrum $n_{\mathbf{k}}^B$. One should note that we define the particle spectrum so that $N = \int 4\pi k^2 n_k dk$, where N is the mean particle density. A different convention was used in [9]: the particle spectrum was defined in the way that $N = \frac{1}{(2\pi)^3} \int 4\pi k^2 n_k dk$. This difference in notation has to be taken into account when dealing with the experimental data.

In the acoustic limit, $\psi_{\mathbf{k}} \rightarrow \frac{1}{2\xi\sqrt{c_s k}}(a_{\mathbf{k}} - a_{-\mathbf{k}}^*)$. Assuming that $a_{\mathbf{k}}$ and $a_{-\mathbf{k}}$ have the same amplitude and independent random phases (consistent with the so-called random amplitude and phases assumption in WWT plus the isotropy assumption), one gets

$$n_{\mathbf{k}} = \frac{\rho_0}{2m\xi^2 c_s k} n_{\mathbf{k}}^B. \quad (19)$$

In the short-wave limit, we have $\psi_{\mathbf{k}} \rightarrow \frac{1}{\sqrt{2c_s \xi}} a_{\mathbf{k}}$, which results in

$$n_{\mathbf{k}} = \frac{\rho_0}{\sqrt{2} m c_s \xi} n_{\mathbf{k}}^B. \quad (20)$$

To obtain isotropic WKE, we calculate the angular average of the Dirac- δ function of wavevectors in Eq. (7), $\Delta = \langle \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \rangle$ (see Supplementary Fig. 1), yielding $\Delta = \frac{2\pi}{k k_1 k_2}$. Remarkably, we discover that for the degenerate resonant manifold of non-dispersive sound waves, $\Delta_{\text{nondisp}} = \frac{\pi}{k k_1 k_2}$, which regularizes the 3D non-dispersive isotropic WKE of acoustic turbulence. Nonetheless, to make sense of the WKE, one should take the limit of the large box first, followed by the angle average, and finally the large time limit (small nonlinearity). The resulting isotropic 3D acoustic WKE has the same form as Eq. (10), except that the prefactor is now $2\pi^2 V_0^2 / c_s$. In Supplementary Note 1, we present a revised, rigorous derivation of the WKE Eq. (9), and the parametrisation of the resonant manifold defined by Eq. (5).

We seek for stationary power-law solutions of wave-kinetic equations using the standard WTT procedure [1]. We provide all the technical information in Supplementary Note 2, including the

calculation and locality analysis, while the main text outlines the essential steps. Supplementary Fig. 2 presents the dimensionless collision integrals $I(x)$ in their convergence windows. Note that for the non-dispersive acoustic case, we have $P(k) = 8\pi^3 V_0^2 A^2 \frac{I(x)}{2x-9} k^{9-2x}$, in which the non-dimensional collision integral $I(x)$ is the same as the one for dispersive acoustic waves, and obtain

$$C_{1,\text{nondisp}} = \frac{\sqrt{2}}{\sqrt{3\pi(4\ln 2 + \pi - 1)}}. \quad (21)$$

run	model	N_r	$k_{\min}$	$k_{\max}$	k_f	s	
1	Bogoliubov WKE	32768	$1 \times 10^{-2}/\xi$	$200/\xi$	0	4	
2	acoustic WKE	8196	1×10^{-4}	1	0.01	2	
run	model	f_0		Δk_f	β	k_d	
1	Bogoliubov WKE	1.032×10^{14}		1.5×10^{-2}	1	80	
2	acoustic WKE	2.419×10^{16}		0.028	1	0.6	
run	model	ξ	N_p	α	k_R	f_w^2	γ
3	GPE	$1.25\Delta x$	960	3	260	10^{-9}	-1
4	GPE	$15\Delta x$	576	4	300	10^{-4}	-1
5	GPE	$1.25\Delta x$	384	3	128	—	—
6	GPE	$10\Delta x$	384	4	200	—	—

Table 1 Parameters used in the simulations.

Parameters for Gross-Pitaevskii equation (GPE) and wave-kinetic equation (WKE) simulations. For the WKE runs, N_r is the number of grid points, $k_{\min}$ and $k_{\max}$ are the minimum and maximum wave numbers of the computational domain, k_f is the forcing wave number, s is the forcing exponent, f_0 is the forcing amplitude, Δk_f is the forcing bandwidth, and β and k_d are the dissipation exponent and dissipation wave number. For the GPE runs, ξ is the healing length, N_p is the number of collocation points per spatial direction, α and k_R are the hyper-viscous dissipation exponent and reference wave number, and f_w^2 and γ are the amplitude and correlation-time parameters of the stochastic forcing.

Numerical simulation methods

We perform GPE simulations using the standard massively-parallel pseudo-spectral code FROST [21] with a fourth-order Exponential Runge-Kutta temporal scheme [19]. We use grids of N_p^3 collocation points so that the grid length is $\Delta x = L/N_p$. The forced-dissipated GPE Eq. (4) is used for the GPE runs 3 and 4, with stochastic forcing term supported on a narrow band ($1 \leq k \leq 3$) and obeying the Ornstein-Uhlenbeck process $d\mathcal{F}_{\mathbf{k}}(t) = -\gamma \hat{\Psi}_{\mathbf{k}} dt + f_w dW_{\mathbf{k}}$ where $W_{\mathbf{k}}$ is the Wiener process. The parameters γ and f_0 control the correlation time and the amplitude of the forcing, respectively. In addition, the condensate amplitude (at $k = 0$) is kept constant during evolution. Dissipation is implemented through the

operator $\mathcal{D}$ in Eq. 4. In Fourier space, its action is $\hat{\mathcal{D}}(\mathbf{k}) = D_k \hat{\Psi}_{\mathbf{k}}$, where $D_k = \left(\frac{k}{k_R}\right)^{2\alpha}$, representing a smooth hyper-viscous damping acting at small scales. Finally, the k -space energy fluxes $P(k)$ are computed directly using the GPE Eq. (4) (see Supplemental Materials of [12]). We verify that the simulated WT is in a weakly nonlinear regime by measuring spatio-temporal spectra of $\Psi(\mathbf{x}, t)$ and confirming that they are narrowly concentrated around the dispersion curve Eq. (1) (see Supplementary Note 3 and Supplementary Fig. 3).

To simulate BEC with a potential trap, we set a box trap and release the constraint on the $k = 0$ mode. The potential term reads $V(\mathbf{x}) = 0$ for coordinates satisfying $|x|, |y|, |z| \leq L_{\text{trap}}$, and $V_{\text{box}}(\mathbf{x}) = U_D$ otherwise. $U_D = 1000\mu$ is the value we select for the GPE runs 5 and 6. $\mu = mc_s^2$ is the chemical potential. We obtain the ground state using the imaginary-time scheme with this box trap. We then add energy by replacing the forcing term $i\mathcal{F}$ in Eq. (4) by a shaking potential term $V_{\text{osc}}(\mathbf{x}, t)$, namely, $i\mathcal{F} = V_{\text{osc}}(\mathbf{x}, t) = Ax \sin(\omega_{\text{res}} t) + Ay \cos(\frac{\sqrt{5}}{2} \omega_{\text{res}} t)$. Note that we shake the trap in both x and y directions to isotropize the system, which is particularly necessary to obtain acoustic waves at relatively large scales. The factor $\sqrt{5}/2$ is chosen so that the ratio of the shaking frequencies in the x and y directions is irrational. This makes the drivings in the two directions effectively independent, while keeping the frequencies as close as possible. The shaking amplitude is set as $A = 0.5\mu/L_{\text{trap}}$ and the frequency is set as the Bogoliubov frequency of the trap, which is $\omega_{\text{res}} = c_s k_{\text{trap}} \sqrt{1 + (k_{\text{trap}} \xi)^2}/2$. Dissipation is set in the same way as in the periodic case, but the condensate amplitude is not fixed. The energy flux is estimated under the stationary-state assumption from the total dissipation by $P_0 = 2 \int D_k E(k) dk$. This provides a practical flux normalisation for the weak-wave-turbulence-dominated regime considered here, especially in the high- k range where the dissipation acts and the nonlinear contribution is expected to remain very weak. The density, sound speed, and healing length used in Fig. 1(a) for runs 5 and 6 are the effective values inside the trap when the system gets stationary.

We simulate isotropic WKE using the code WavKinS [22]. Similarly to the GPE, we also include a forcing term of the form

$f_k = f_0 k^s e^{-(k-k_f)^2/\Delta k_f^2}$ and dissipation term $-(k/k_d)^\beta n_k$ in the right-hand side of WKE Eq. (9) and Eq. (10). The code uses a logarithmic mesh $\{k_i = C\lambda^i, i = 1, \dots, N_r\}$, where λ is chosen in order to span the integration domain $[k_{\min}, k_{\max}]$. Inter-mesh values are obtained using a linear interpolation that ensures positivity. The collision integral is a 1D integral obtained by making explicit use of the parametrisation of the resonant manifold provided in Supplementary Note 1. Integrals are performed using the trapezoidal rule, after performing a change of variable in the following manner $\int_{k_i}^{k_{i+1}} G(p) dp \approx \frac{\ln \lambda}{2} (G(k_{i+1})k_{i+1} + G(k_i)k_i)$. We make an approximation of $\int_0^{k_1} G(p) dp \approx k_1 G(k_1)$ for the integral on the first bin. Finally, we use a Runge-Kutta-2 time marching method, with an implicit scheme for the dissipative term.

All essential numerical parameters for the WKE and GPE runs are given in Table 1. The parameters in each run were chosen to maximize the inertial (non-dissipative) range of k in the regime at which this specific run is focused: large-scale acoustic, small-scale Bogoliubov, or the regime containing both large- and small-scale WT.

Equation of state predicted by four-wave and three-wave theories and fitting method

In reference [8], the constant energy flux solution in the 4-wave regime was derived analytically and confirmed numerically. Its log-corrected spectrum reads

$$n_k = C_d (\epsilon \hbar / g^2)^{1/3} k^{-3} \ln^{-1/3}(k/k_f). \quad (22)$$

Substituting $g = 4\pi \hbar^2 a_s / m$ into the above equation, one gets the equation of states of Eq. (13).

We recall the relationship between the particle spectrum and the wave-action spectrum for short Bogoliubov waves Eq. (20), and substitute $n_k^B = \frac{E(k)}{4\pi k^2 \omega_{\text{short}}(k)}$, the KZ solution Eq. (3), and $P_0 = \epsilon / \rho_0$ into it. This gives

$$n_k = \frac{C_2}{4\pi} \frac{\rho_0^{1/2} \xi^{1/2}}{m c_s^{3/2}} \epsilon^{1/2} k^{-3}. \quad (23)$$

Then we express the parameters c_s , ξ and ρ_0 in terms of $\hbar$, m , a_s and N by the definitions $\xi = \sqrt{\frac{\hbar^2}{2g\rho_0}}$, $c_s = \sqrt{\frac{g\rho_0}{m^2}}$, $g = 4\pi \hbar^2 a_s / m$, and $\rho_0 = mN$. Finally, one gets

$$n_k = \frac{C_2}{2^{17/4} \pi^2} \frac{m \epsilon^{1/2}}{\hbar^{3/2} N^{1/2} a_s} k^{-3}, \quad (24)$$

and consequently the EoS Eq. (15).

In generating Fig. 2, each experimental spectrum is fitted using the 4-wave prediction $n_k = n_{4w}^{\text{exp}} k^{-3} \ln^{-\frac{1}{3}}\left(\frac{k}{k_f}\right)$ and the 3-wave prediction $n_k = n_{3w}^{\text{exp}} k^{-3}$, respectively, yielding the experimentally obtained spectrum amplitude n_{4w}^{exp} and n_{3w}^{exp} . To determine the most suitable candidate between these two models, the theoretical amplitudes n_{4w} and n_{3w} are calculated using Eq. (13) and Eq. (15). The model is then selected based on the computed ratio $n_{4w}^{\text{exp}}/n_{4w}$ and $n_{3w}^{\text{exp}}/n_{3w}$, prioritizing the one whose ratio is closer to unity. An explicit example in which the 4-wave prediction is selected is shown in Fig. 3, corresponding to the left-most data point in Fig. 2(a). Note that the vertical-axis scale in Fig. 2(a) differs from that in Fig. 4(a) of [9], because a different convention for $n(k)$ is used here, differing by a factor of $1/(8\pi^3)$, as explained earlier in the Methods.

Declarations

Author contribution. Y.Z., G.K., and S.N. designed research; performed research; and wrote the paper.

Conflict of interest. The authors declare no competing interest.

Data Availability. The numerical data underlying Figs. 1–3 and Supplementary Fig. 3 are provided as Supplementary Data with this paper. The raw numerical simulation data generated in this study are stored on institutional storage facilities and are not publicly deposited because of their large size, but are available from the corresponding author upon reasonable request. The experimental data from Ref. [9] were obtained from the Apollo repository [23].

Code Availability. The Gross-Pitaevskii equation simulations were performed using the pseudo-spectral code FROST, which is

available from the corresponding author upon reasonable request. The wave-kinetic equation simulations were performed using the open-source code WavKinS, which is available at <https://github.com/gkrstulovic/WavKinS.jl> [22]. The scripts used for data post-processing and figure generation are available from the corresponding author upon reasonable request.

Funding. This work was funded by the Simons Foundation Collaboration grant Wave Turbulence (Award ID 651471). Y.Z. and S.N. acknowledge support from ANR through the project VORTECS (Grant No. ANR-22-CE30-0011).

Acknowledgements. This work was provided with HPC and storage resources of IDRIS and CINES under the allocations A0152A14637 and A0172A14637 made by GENCI on the Jean Zay (SKL partition) and Adastra (GENOA partition) supercomputers. This work was granted access to the OPAL infrastructure from Universit   C  te d’Azur, supported by the French government through the UCAJEDI Investments in the Future project managed by the National Research Agency (ANR) under Reference No. ANR-15-IDEX-01.

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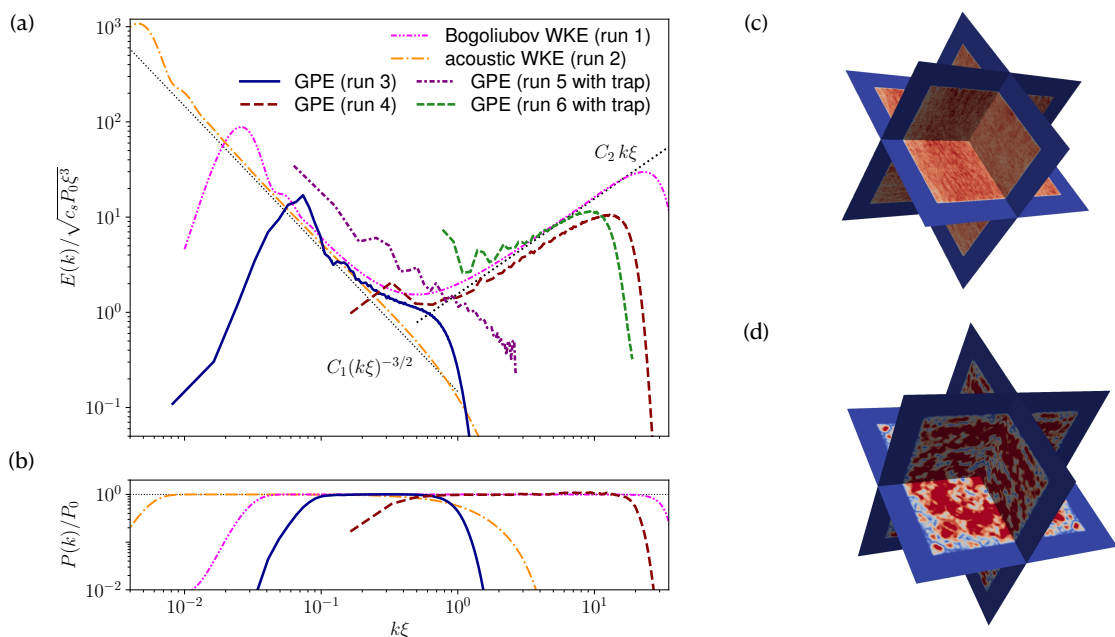


Fig. 1 Numerical validation of the Kolmogorov-Zakharov type spectra for Bogoliubov waves. (a) Dimensionless energy spectra $E(k)/\sqrt{c_s P_0 \xi^3}$ obtained from wave-kinetic equation (WKE) and Gross-Pitaevskii equation (GPE) simulations, plotted versus the dimensionless wave number $k\xi$ and compared with the theoretical predictions Eq. (2) and Eq. (3). Here $E(k)$ is the energy spectrum, k is the wave number, c_s is the sound speed, P_0 is the constant energy flux measured in the inertial range, and ξ is the healing length. Run 1 is performed using the Bogoliubov WKE Eq. (9) and contains both the acoustic, or long-wave, and short-wave regimes, whereas run 2 uses the three-dimensional acoustic WKE Eq. (10). The GPE runs 3 and 4 are obtained for homogeneous cases with stochastic forcing, and runs 5 and 6 are obtained with a shaken potential trap. All the GPE runs are optimised such that the inertial range extends through the scales of interest. (b) Corresponding energy flux spectra $P(k)$ normalised by P_0 for runs 1–4. (c) Snapshot of density for GPE run 5 to verify the prediction in the acoustic limit. (d) Snapshot of density for GPE run 6 to verify the prediction in the short-wave limit.

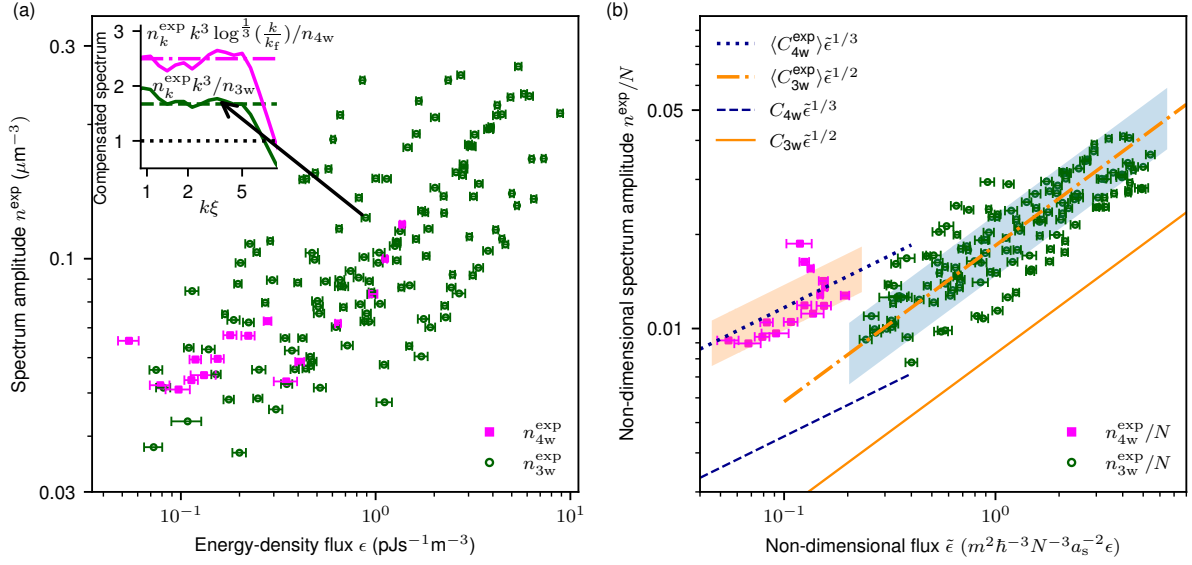


Fig. 2 Reconstructed equation of state from experimental spectra. The equation of state (EoS) is regenerated from the experimental data of Ref. [9]. (a) Particle spectrum amplitude n^{exp} versus energy flux ϵ , where ϵ equals the energy dissipation rate per unit volume. The amplitude n^{exp} is taken either as n_{4w}^{exp} , shown in magenta and fitted by the 4-wave prediction Eq. (13), or as n_{3w}^{exp} , shown in green and fitted by the 3-wave prediction Eq. (15). The fitting procedure is indicated by the inset of (a): each experimental particle spectrum n_k^{exp} , where k is the wave number, is compensated by $k^3 \ln^{1/3}(\frac{k}{k_f})$ and k^3 , respectively, where k_f is the forcing wave number. The fitted amplitudes n_{4w}^{exp} and n_{3w}^{exp} are calculated by averaging the corresponding compensated spectrum on a plateau range. We take n^{exp} , n_{4w}^{exp} or n_{3w}^{exp} depending on which is closer to the corresponding theoretical prediction, n_{4w} or n_{3w} , calculated from Eqs. (13) and (15), respectively. (b) Non-dimensional particle spectrum amplitude n^{exp}/N versus non-dimensional flux $\tilde{\epsilon} = m^2\hbar^{-3}a_s^{-2}N^{-3}\epsilon$, where N is the mean particle density, m is the atom mass, a_s is the s-wave scattering length characterising interaction strength, and $\hbar$ is the reduced Planck constant. Theoretical predictions $C_{4w}\tilde{\epsilon}^{1/3}$ and $C_{3w}\tilde{\epsilon}^{1/2}$, together with the fitted curves obtained by replacing C_{4w} and C_{3w} by the mean fitted constants $\langle C_{4w}^{\text{exp}} \rangle$ and $\langle C_{3w}^{\text{exp}} \rangle$, are superimposed. The shaded bands around the latter two curves indicate one standard deviation of the logarithmic deviations of the data points from the fitted lines, computed separately from 16 data points assigned to the 4-wave branch and 136 data points assigned to the 3-wave branch. The horizontal error bars indicate the flux uncertainties provided in the original experimental data; no additional statistical averaging was performed in this work to obtain these error bars.

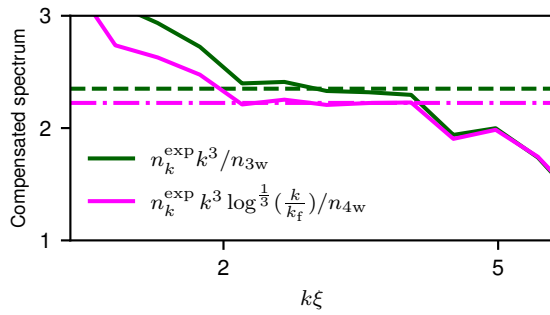


Fig. 3 Example selecting the four-wave prediction.

Experimental particle spectrum n_k^{exp} compensated by k^3 and $k^3 \ln^{1/3}(\frac{k}{k_f})$, and normalised by the corresponding 3-wave and 4-wave theoretical spectral amplitudes n_{3w} and n_{4w} , respectively, where k is the wave number, ξ is the healing length, and k_f is the forcing wave number. The spectrum corresponds to the leftmost point in Fig. 2(a).